KEY

Exam 2

MATH 221-2

April 1, 2015 Chapters 1-5

/100

1. (19 pts, 1 pt each) Circle **T** or **F** indicating whether each of the following statements are True or False. You do not need to justify your answers.

(a) **T** **F** A 3 dimensional vector space must contain the 3×1 zero vector $\mathbf{0}$.

(b) **T** **F** The set of vectors $\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$ in $\mathbb{R}^3$ with $b_1 = b_2$ is a vector space.

(c) **T** **F** If A is a randomly generated 10×13 matrix, then it is almost certain that A has 13 pivots.

(d) **T** **F** If $\boldsymbol{x} \in N(A)$, then $A\boldsymbol{x} = \mathbf{0}$.

(e) **T** **F** The rank of a matrix is the number of columns (n) minus the number of free variables.

$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$

(f) **T** **F** The subspace of $\mathbb{R}^2$ spanned by all vectors with positive components is the positive quadrant of $\mathbb{R}^2$.

(g) **T** **F** The vector $\boldsymbol{c}$ is in $C(A^T)$ if $A^T\boldsymbol{y} = \boldsymbol{c}$ has a solution.

(h) **T** **F** If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution and if $A^T\boldsymbol{y} = \mathbf{0}$, then $\boldsymbol{b} \cdot \boldsymbol{y} = 0$.

(i) **T** **F** For any matrix A, $C(A)$ and $N(A^T)$ are orthogonal vector spaces.

(j) **T** **F** If $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal, then $\boldsymbol{x} \cdot \boldsymbol{y} = 1$.

(k) **T** **F** If $A = \begin{bmatrix} | & | & & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_n \\ | & | & & | \end{bmatrix}$ then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ is a basis for $C(A)$.

(l) **T** **F** If A is a 15×14 matrix then A^{-1} is a 14×15 matrix.

(m) **T** **F** If A is a 3×3 matrix and $\dim(C(A)) = 3$ then A is invertible.

(n) **T** **F** If the columns of A are linearly independent, then there is only one least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$.

(o) **T** **F** If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$ then $\dim(C(A)) = 2$.

(p) **T** **F** Any set of n linearly independent vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

(q) **T** **F** If the columns of A are linearly independent, then there is always at least 1 solution to $A\boldsymbol{x} = \boldsymbol{b}$.

(r) **T** **F** If A is a 3×2 matrix and $A\boldsymbol{x} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$ has only one solution then $N(A) = \{\mathbf{0}\}$.

2. (12 pts) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$. Find the least squares solution to $Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. SHOW YOUR WORK!

$$A^T A \underline{x} = A^T \underline{b} \iff \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\iff \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

$$\Rightarrow \boxed{\begin{matrix} x = 3 \\ y = 1 \end{matrix}}$$

3. (a) (8 pts) Let $A = \begin{bmatrix} 1 & 2 & 3 & 7 \\ 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 91 \end{bmatrix}$. Show that A is not invertible (using whatever method of your choice). Show your work.

Applying Gauss-Elimination: $R_2 \leftarrow -R_1 + R_2$

$R_4 \leftarrow -R_1 + R_4$

$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 84 \end{bmatrix}$ shows that there are only 2 pivots $\Rightarrow$ A is not invertible!!

(b) (6 pts) Based on your answer to (a), for any RHS $b \in C(A)$, how many solutions do you expect to $Ax = b$? Explain.

Because A is not invertible, then $\dim(C(A)) = \dim(R(A))$ is less than 4 $\Rightarrow$ A has null spaces $N(A)$ and $N(A^T)$

$\Rightarrow$ By Fredholm Alternative, there are either 0 or ∞ of solutions for any RHS.

4. Let $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 6 & 6 \end{pmatrix}$.

(a) (13 pts) Solve the system of linear equations

$$\begin{array}{ccccccc} x & + & 3y & + & z & = & 1 \\ 2x & + & 6y & + & 6z & = & 10 \end{array}.$$

Write the complete solution as a particular plus a null (or homogeneous) solution. SHOW YOUR WORK.

the augmented matrix is $\begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 6 & 6 & 10 \end{bmatrix}$

Applying Gauss Elimination: $R_2 \leftarrow -2R_1 + R_2$, get:

$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & 4 & 8 \end{bmatrix}$. thus $4z = 8 \Rightarrow \boxed{z = 2}$.

Let $\boxed{y \text{ be free}}$. So $x = 1 - 3y - z$

$\boxed{x = -1 - 3y}$

$$\therefore \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -1-3y \\ y \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{x}_p \quad + \quad \underline{x}_n$

(b) (6 pts) Based on the work you did in #4a, give a basis for $C(A)$. Justify your answer.

ref (A) has pivots in columns 1 & 3.

$\therefore C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \end{bmatrix}\right)$.

(c) (6 pts) Based on your answer to #4a, give a basis for $N(A)$. Justify your answer.

the basis for $N(A)$ is given by $\underline{x}_n$:

$N(A) = \text{span}\left(\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}\right)$.

(d) (6 pts) Based on your answer to #4a, is $\begin{bmatrix} 1 \\ 10 \end{bmatrix} \in C(A)$? Explain why your answer is correct.

Yes, because we were able to solve

$$A\underline{x} = \underline{b}.$$

(e) (6 pts) For any non-zero $\boldsymbol{b} \in \mathbb{R}^2$ that one might choose, explain why finding only 1 solution to $A\boldsymbol{x} = \boldsymbol{b}$ is impossible.

Because $N(A)$ is 1-dimensional always generates an infinite # of solutions via $\underline{x}_n$ as in #4a.

(f) (6 pts) For any $\boldsymbol{b} \in \mathbb{R}^2$ that one might choose, explain why finding no solutions to $A\boldsymbol{x} = \boldsymbol{b}$ is impossible.

Because $C(A) = \mathbb{R}^2$ by #4b, all RHS $\underline{b} \in \mathbb{R}^2$ have a solution to

$$A\underline{x} = \underline{b}$$

5. (12 pts) For any $m \times n$ matrix A, prove that $N(A)$ is a vector space. Explicitly state the conditions that you are checking, and justify each step in your proof.

Closure wrt addition

(A1) Let $\underline{u}, \underline{v} \in N(A)$. Is $\underline{u} + \underline{v} \in N(A)$?

Consider: $A(\underline{u} + \underline{v}) = A\underline{u} + A\underline{v}$ by distributive prop.

$= \underline{0} + \underline{0}$ because $\underline{u}, \underline{v} \in N(A)$

$= \underline{0}$.

$\therefore$ $\underline{u} + \underline{v} \in N(A)$.

(M1) Closure wrt scalar multiplication. Let $\underline{u} \in N(A)$. Is $c\underline{u} \in N(A)$? any $c \in \mathbb{R}$

Consider $A(c\underline{u}) = cA\underline{u}$ by associativity

$= c\,\underline{0}$ because $\underline{u} \in N(A)$.

$= \underline{0}$

$\therefore$ $c\underline{u} \in N(A)$.