KEY.

Final Exam 3

MATH 221-02

May 1, 2014

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1. (21 pts) Circle T or F indicating whether each of the following statements are True or False. You do not need to justify your answer.

(a) T F If $\vec{u} \cdot \vec{w} = 0$, then $\vec{u}$ and $\vec{w}$ are parallel.

(b) T F If A, B and C are square matrices and if $A = BC$, then $\det(A) = \det(B)\det(C)$.

(c) T F The matrix $\begin{bmatrix} -1 & -4 & 0 \\ 2 & 3 & 3 \\ 0 & 2 & 0 \end{bmatrix}$ is invertible.

(d) T F $\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$.

(e) T F If $A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}$, then $\mathcal{N}(A) \neq \{0\}$.

(f) T F For any matrix A, $\dim(\mathcal{R}_A) = \dim(\mathcal{C}_A)$.

(g) T F If A and B are invertible then $(AB)^{-1} = A^{-1}B^{-1}$.

(h) T F If A is square and $\det(A) = 0$, then $A\boldsymbol{x} = \boldsymbol{b}$ has no solutions.

(i) T F If B is a 700×500 matrix, then B^{-1} is a 700×500 matrix.

(j) T F If $A\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$ then A is not invertible.

(k) T F If A is square and $A\boldsymbol{x} = \boldsymbol{b}$ has only one solution, then A is invertible.

(l) T F When solving $A\boldsymbol{x} = \boldsymbol{b}$, the number of free variables is equal to $\dim(\mathcal{N}(\mathcal{A}))$.

(m) T F The vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is in the nullspace of $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

(n) T F The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

(o) T F Every square matrix has at least one eigenvalue and eigenvector.

(p) T F If A is symmetric then A is diagonalizable.

(q) T F The matrix A^TA is diagonlizable.

(r) T F If A is invertible then A is diagonalizable.

(s) T F If $\lambda = 0$ is an eigenvalue of A, then the linearly independent eigenvectors associated with $\lambda = 0$ are a basis for $\mathcal{N}(A)$.

(t) T F The geometric multiplicity of an eigenvalue λ of A is $\dim(\mathcal{N}(A - \lambda I))$.

(u) T F If $A\vec{u} = 5\vec{u}$ and $A\vec{v} = -5\vec{v}$ then $\vec{u} + \vec{v}$ is an eigenvector of A.

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2. (12 pts) In each of the following, circle an appropriate statement from the options *statement1/statement2.* You do not need to justify your answers.

Assume that A is $n \times n$ and that the matrix system $Ax = b$ has <u>no</u> solutions.

(a) rank(A) is/(is not) equal to n.
(b) A is invertible/(not invertible).
(c) $Ax = b$ is consistent/(inconsistent.
(d) det(A)(is)/(is not) equal to zero.
(e) b is/(is not) a linear combination of the columns of A.
(f) The columns of A are linearly (dependent)/independent.
(g) $Ax = b$ has /(does not have) exactly one least squares solution.
(h) When performing Gauss-Elimination to put the matrix A into row echelon form, there (are)/(are not) zero pivot(s).
(i) A (has)/ (does not have) a zero eigenvalue.
(j) A (has)/ (does not have) non-zero homogeneous solutions.
(k) $\mathcal{N}(A)$ is/(is not) equal to $\{0\}$.
(l) $\mathcal{N}(A^T)$ is/(is not) equal to $\{0\}$.

3. (1~~4~~5 pts) Solve the system of ODEs $\begin{array}{rcrcr} \dot{x} & = & x & + & 2y \\ \dot{y} & = & -2x & + & y \end{array}$. SHOW YOUR WORK! *Hint:* In general, the solution can be written as $c_1 e^{\lambda_1 t} p_1 + c_2 e^{\lambda_2 t} p_2$; or if appropriate, write solution as

$$c_1 e^{\alpha t} (a \cos(\beta t) - b \sin(\beta t)) + c_2 e^{\alpha t} (a \cos(\beta t) + b \sin(\beta t)).$$

Also, the quadratic formula might be handy: $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Need to find eigenvalues first: solve

$$\det(A - \lambda I) = 0$$

where $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, giving

$$\det \begin{pmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 4 = \lambda^2 - 2\lambda + 5 = 0$$

$$\therefore \lambda = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2} = 1 \pm \frac{\sqrt{-16}}{2} = \boxed{1 \pm 2i}$$

So $\alpha = 1$, $\beta = 2$.

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Next, the eigenvectors; solve

$(A-\lambda I)\underline{p}=\underline{0}$, which has augmented matrix

$$\begin{bmatrix} 1-(1+2i) & 2 & 0 \\ -2 & 1-(1+2i) & 0 \end{bmatrix} = \begin{bmatrix} -2i & 2 & 0 \\ -2 & -2i & 0 \end{bmatrix}$$

$R_2 \leftarrow iR_1 + R_2$ gives:

$$\begin{bmatrix} -2i & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Letting p_2 be free gives solution

$$\underline{p} = \begin{bmatrix} \frac{-2}{-2i}p_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} -ip_2 \\ p_2 \end{bmatrix} = p_2\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} i\right]$$

$\left(\text{or } \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} i\right)$

So $\underline{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Plugging into formula:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^t\left[c_1\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin 2t\right) + c_2\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos 2t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin 2t\right)\right]$$

4. (2 pts) The qualitative behavior of this system is (choose one of A-E):
A. an exponentially increasing function; B. an exponentially decreasing function;
C. an oscillation with decreasing amplitude; D. an oscillation with increasing amplitude;
E. An oscillation with constant amplitude.

5. (14 pts) You are solving a system of 5 equations with 3 unknowns. You write out the system as $A\vec{x} = \vec{b}$. You apply Gauss elimination to the coefficient matrix A and get the following reduced row echelon form,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Complete the following statements (you do not need to justify your answers):

(a) (1 pt) The dimension of the domain of A is 3.

(b) (1 pt) $\dim(\mathcal{C}_A) =$ 2.

(c) (1 pt) $\dim(\mathcal{R}_A) =$ 2.

(d) (1 pt) $\dim(\mathcal{N}(A)) =$ 1.

(e) (1 pt) $\dim(\mathcal{N}(A^T)) =$ 3.

(f) (2 pts) If $\boldsymbol{b}$ is in $\mathcal{C}_A$ then the number of solutions to the system $A\boldsymbol{x} = \boldsymbol{b}$ is ∞.

(g) (2 pts) If $\boldsymbol{b}$ is not in $\mathcal{C}_A$ then the number of solutions to $A\boldsymbol{x} = \boldsymbol{b}$ is 0.

(h) (2 pts) Give a basis for $\mathcal{R}_A$.

It's the rows with non-zero pivots:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(i) (4 pts) Give a basis for $\mathcal{N}(A)$.

It's the vector $\perp$ to $\mathcal{R}_A$: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Or find it explicitly: augmented matrix for $A\underline{x} = \underline{0}$ in rref:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

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6. (10 points) The 5×5 matrix A has the characteristic equation $\rho_A(\lambda) = (\lambda - 3)^2(\lambda + 4)^3$.

   (2 points each) The eigenvalues of A are (order them $\lambda_1 < \lambda_2$) (a) $\lambda_1 =$ __-4__,

   and (b) $\lambda_2 =$ __3__. Their algebraic multiplicities are (c) __3__ and

   (d) __2__, respectively. A is diagonalizable if the geometric multiplicity of λ_1

   is (e) __3__ and the geometric multiplicity of λ_2 is (f) __2__.

7. You collect data on Bozeman water usage in the years 2011, 2012 and 2013 from the facilities that provide our awesome drinking water. The covariance matrix for these 3 variables (i.e., water usage in each of the years 2011, 2012 and 2013) is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}.$$

Recall that the 1st row and column of C correspond to the 1st variable, in this case 2011 water usage; the 2nd row and column correspond to the 2nd variable, 2012 water usage; and the 3rd row and column correspond to the 3rd variable, 2013 water usage.

   (a) (6 pts) Give the eigenvalues of A. SHOW YOUR WORK!

   $\rho_A = \det(A - \lambda I) = (1-\lambda)(2-\lambda)(10-\lambda)$

   The roots of ρ_A are the eigenvalues,

   $\lambda = 1, 2, 10$

   (b) (6 pts) Give the single "best" variable (i.e., the eigenvector that corresponds to the first principle component) that describes the water usage in 2011, 2012, 2013. SHOW YOUR WORK!

   the 1st PC is eigenvector corresponding to $\lambda = 10$, found via solving $(A - 10I)\underline{p} = \underline{0}$. The augmented matrix is:

   $$\begin{pmatrix} -9 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{p} = \begin{pmatrix} 0 \\ 0 \\ p_3 \end{pmatrix} = p_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   $\therefore$ the best variable is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, i.e, the 2013 data.

   (c) (3 pts) What proportion of the variability on the water usage data from 2011-2013 is explained by the "best variable" you found in #7b. SHOW YOUR WORK!

   $$\frac{10}{1+2+10} = \frac{10}{13}.$$

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8. A biologist with Montana Fish and Wildlife wants to estimate the number of elk that visit Colby Creek in the Bridger Mountains. When monitoring the drainage for one 24 hour period, $y_1 = 5$ elk are observed. When monitoring over a 24 hour period a few weeks later, $y_2 = 2$ elk are observed. And when monitoring another time, $y_3 = 10$ elk are observed. The biologist wants to fit the "best constant" to the data; that is, he wants to fit the model

$$y = c.$$

(a) (3 pts) What is the equation $A\boldsymbol{x} = \boldsymbol{b}$ to be used to find c? Be sure to identify A, $\boldsymbol{x}$, and $\boldsymbol{b}$.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} c \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix}$$

(b) (6 pts) Find the least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$. SHOW YOUR WORK!

The least squares solution is

$$\underline{x}_{ls} = (A^T A)^{-1} A^T \underline{b}.$$

$$= \left[\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix}$$

$$= 3^{-1}(5 + 2 + 10)$$

$$= 17/3.$$

$$= 5\,2/3.$$

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