2. Let $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.

$\boxed{2 \text{ pts}}$ (a) Find a basis for $N(A)$. SHOW YOUR WORK!

Solving $A\underline{x} = \underline{0}$, get augmented matrix $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$

If $\underline{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, then let $\boxed{c \,\&\, d \text{ be free}}$ then

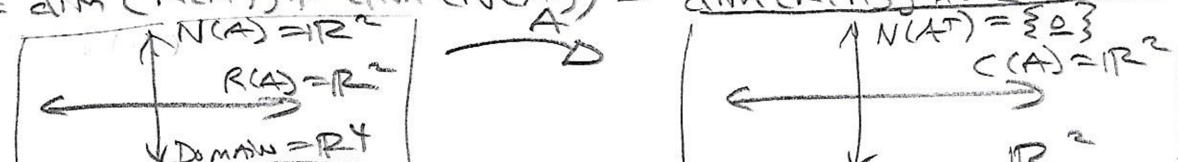
$\boxed{b = -c - d}$ and $a = -2b - d = -2(-c-d) - d = \boxed{2c + d}$

So $\underline{x} = \begin{pmatrix} 2c+d \\ -c-d \\ c \\ d \end{pmatrix} = c\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$ Basis for $N(A)$ is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\boxed{2 \text{ pts}}$ (b) Write out the "blob diagram" that clearly shows where the domain of A, $C(A)$, $R(A)$, $N(A^T)$, and $N(A)$ reside, and indicate the dimensionality of each of these 5 vector spaces.

DOMAIN $= \mathbb{R}^4 = \dim(R(A)) + \dim(N(A)) = \dim(R(A)) + 2$

So $R(A) = \mathbb{R}^2 \Rightarrow$



$\boxed{1 \text{ pt}}$ (c) When solving $Ax = b$ for any RHS $b \in \Re^3$, use the Fredholm Alternative to argue how many solutions you expect for get. *Hint: there is only one possibility.*

Is $b \in C(A)$? $\xrightarrow[\text{because}]{\text{Always Yes}}$ Is there a $N(A)$? $\xrightarrow{\text{Yes}}$ There are always an ∞ number of solutions

$N(A^T) = \{\underline{0}\}$

$\boxed{1 \text{ pt}}$ (d) Try to apply the method described in class to find the 2×2 projection matrix that projects 2×1 vectors x onto $C(A)$. You will find that that the method fails. At which point does the method fail? SHOW YOUR WORK!

When trying to calculate $P = A(A^TA)^{-1}A^T$, $(A^TA)^{-1}$ cannot be calculated because $A^TA = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 5 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \end{bmatrix}$

Applying GE to 1st column

$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ $\xrightarrow{\text{Now applying GE to second column}}$ $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \det(A^TA) = 0$.

$\Rightarrow \boxed{A^TA \text{ is NOT invertible}}$

$\boxed{1 \text{ pt}}$ (e) What is the difference in the 4 fundamental spaces for the A matrix in #1c (where you could find the projection matrix) and the A matrix in #2d that could explain why you could not find the projection matrix in #2d?

A in #1 has $N(A) = \{\underline{0}\}$, A in #2 has $N(A) = \mathbb{R}^2$