(KEY)

Exam 1

STAT422 — *February 18, 2011*

Instructions: This exam is worth 100 points. SHOW ALL WORK! No work, no credit.

1. (22 pts) Circle True or False to indicate the validity of the following statements. You do not need to justify your answer. In all of the following, assume that $X_1, ..., X_n$ is a SRS from some population of interest with mean μ and variance σ^2.

   (a) True or False: $E(\sum_{i=1}^n X_i) = \sum_{i=1}^n E(X_i)$ only when the data X_i are independent.

   (b) True or False: $Var(X_1 + 2X_2) = Var(X_1) + 2Var(X_2)$ only when the data X_i are independent.

   (c) True or False: The sum of χ^2 rvs has a χ^2 distribution when the rvs are independent.

   (d) True or False: If S^2 is the sample variance, then $\lim_{n\to\infty} Var(S^2) = 0$.

   (e) True or False: For the normal distribution $N(\mu, \sigma^2)$, $\bar{X}$ is the MVUE for μ. → minimum variance unbiased estimator

   (f) True or False: $\frac{1}{n}\sum_i (X_i - \bar{X})^2$ is an unbiased estimator for σ^2.

   (g) True or False: Using $p = .5$ maximizes the variance of a binomial random variable.

   (h) True or False: A t statistic with n degrees of freedom is defined as $t = \frac{Z}{\sqrt{U}}$ where $Z \sim N(0, 1)$ and $U \sim \chi^2(n)$.
   → Should be $Z/\sqrt{U/n}$

   (i) True or False: The MOM estimator for $E(X^3)$ is $\frac{1}{n}\left(\sum_{i=1}^n X_i\right)^3$.
   → Chapter 9

   (j) True or False: You measured the temperature on 5 randomly chosen days in January, 2011 in Bozeman. Assuming the data are normal, a 95% CI for the true mean January temperature is $\bar{X} \pm 1.96\frac{S}{\sqrt{n}}$.

   (k) True or False: A 95% confidence interval for $\mu_1 - \mu_2$ is [-6, -1]. Thus, with 95% confidence, $\mu_2 > \mu_1$.

2. (6 pts) State the Central Limit Theorem. Be sure to state all assumptions, and give a proper conclusion.

See theorem 7.4 for the correct answer to this problem.

3. (6 pts) State the Weak Law of Large Numbers. Be sure to state all assumptions, and give a proper conclusion.

Given $X_1, \dots, X_n$ i.i.d. with $EX_i = \mu$ and $VX_i = \sigma < \infty$

$$\bar{X} \xrightarrow{P} \mu$$

i.e. $\bar{X}$ is consistent for μ.

4. (22 pts) On February 5, 2011, the Bozeman Daily Chronicle reported that out of 224 bison captured while attempting to exit Yellowstone National Park, 76 tested positive for brucellosis, a disease which causes animals to abort their young.

(a) (2 pts) Use an unbiased estimator to give an estimate of the true proportion of bison who have brucellosis.

$$\hat{p} = \frac{76}{224} = 0.34$$

(b) (6 pts) Give a 95% CI for the true proportion of Yellowstone bison who have brucellosis.

$$\hat{p} \pm z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.34 \pm 1.96\sqrt{\frac{\frac{76}{224}\left(1-\frac{76}{224}\right)}{224}}$$

$$= [0.277, 0.401]$$

(c) (4 pts) Give a conclusion of this confidence interval in terms of the problem.

We are 95% confident that the true percentage of bison exiting the park that have brucellosis is between 28% and 40%.

(d) (4 pts) The Chronicle reported that "about half of the Park's bison carry the disease." Does your 95% CI refute this claim. Why or why not?

Yes, because the CI does not contain 0.5 (or 50%).

(e) (6 pts) What TWO things must you assume about the sample of bison in order for your conclusions in 4c and 4d to be valid? Do these two assumptions appear to be satisfied? Explain.

The bison used to calculate the CI must be a RS (so data are i.i.d.) of all bison that exit the park. This is extremely unlikely — YNP can only capture bison that use routes monitored by YNP.

5. (18 pts) Suppose that X_1, X_2 are a SRS from $N(\mu, \sigma^2)$.

(a) Give the distribution of $\frac{(Y_1-Y_2)^2}{2\sigma^2}$, and explain why your answer is correct. (*Hint:* What is the distribution of $(Y_1 - Y_2)$?)

$Y_1 - Y_2$ is normal because Y_1 and Y_2 are normal.

$$E(Y_1 - Y_2) = EY_1 - EY_2 = \mu - \mu = 0.$$

$$V(Y_1 - Y_2) = VY_1 + VY_2 = \sigma^2 + \sigma^2 = 2\sigma^2 \text{ because } Y_1 \perp\!\!\!\perp Y_2.$$

$$\therefore\ Y_1 - Y_2 \sim N(0, 2\sigma^2).$$

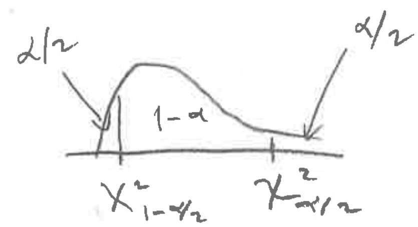
Now standardizing: $(Y_1 - Y_2)/\sqrt{2\sigma^2} \sim N(0,1) \Rightarrow \frac{(Y_1-Y_2)^2}{2\sigma^2} \sim \chi^2(1)$.

(b) Show that $\frac{(Y_1-Y_2)^2}{2\sigma^2}$ is a pivotal quantity for σ^2.

Need: $(Y_1 - Y_2)^2/2\sigma^2$ to be (1) function of data: YES!

(2) Have a known dist. $\perp\!\!\!\perp$ σ^2 : YES, $\chi^2(1)$.

(c) Apply the pivotal method to $\frac{(Y_1-Y_2)^2}{2\sigma^2}$ to construct a CI for σ^2.



$$P\left(\underset{\uparrow \atop \chi^2_{1-\alpha/2}}{a} \le \frac{(Y_1-Y_2)^2}{2\sigma^2} \le \underset{\uparrow \atop \chi^2_{\alpha/2}}{b} \right) = 1 - \alpha$$

$$\Rightarrow P\left(\frac{(Y_1-Y_2)^2}{2b} \le \sigma^2 \le \frac{(Y_1-Y_2)^2}{2a} \right) = 1 - \alpha$$

$$\therefore\ \text{CI is } \left[\frac{(Y_1-Y_2)^2}{2b}, \frac{(Y_1-Y_2)^2}{2a} \right]$$

$\boxed{\text{with } 100 \times (1-\alpha)\% \text{ confidence}}$

6. (12 pts) Let $X_1, ..., X_n$ be a SRS from an exponential distribution with parameter β.

(a) (8 pts) Prove that $\sum_i X_i$ is sufficient for β. Justify each step in your proof.

The exponential likelihood is $L(\beta) = \prod_i \frac{1}{\beta} e^{-x_i/\beta}$

because a SRS

Factoring

$$L(\beta) = \underbrace{\beta^{-n} e^{-\frac{1}{\beta}\Sigma x_i}}_{g(\beta, \Sigma x_i)} \underbrace{1}_{h(\underline{x})} \implies \Sigma x_i \text{ is sufficient for } \beta.$$

(b) (4 pts) Find the MVUE for β. Specify which theorem justifies your answer.

$\bar{X} = \frac{1}{n} \Sigma X_i$ is a function of a sufficient statistic that is unbiased for $EX = \beta$.

By Rao Blackwell, $\bar{X}$ is MVUE.

7. (14 pts) Suppose that $X_1, ..., X_n$ are a SRS from a Poisson distribution with parameter λ.

(a) (10 pts) Find the MLE of λ. Be sure to show that the statistic you find is a maximum.

The Poisson likelihood is $\boxed{L(\lambda) = \prod_i \lambda^{y_i} e^{-\lambda} / y_i! = \lambda^{\Sigma y_i} e^{-n\lambda} / \prod y_i!}$

$\therefore \ln L(\lambda) = \Sigma y_i \ln(\lambda) - n\lambda - \Sigma \log(y_i)$

$\implies \frac{d}{d\lambda} \ln L(\lambda) = \Sigma y_i / \lambda - n$. Setting to 0 shows $\hat{\lambda} = \bar{Y}$ is a critical point.

$\frac{d^2}{d\lambda^2} \ln L(\lambda) = -\frac{\Sigma y_i}{\lambda^2} \leq 0$ because each $y_i \geq 0$.

$\therefore \hat{\lambda}_{MLE} = \bar{Y}$ by 2nd derivative test.

(b) (4 pts) Find the MLE for $\lambda^2 + \lambda$. Justify your answer.

$\hat{\lambda}^2_{MLE} + \hat{\lambda}_{MLE} = \bar{Y}^2 + \bar{Y}$ by Invariance Property of MLEs.