KEY

Exam 1

STAT422 *February 17, 2017*

Instructions: This exam is worth 100 points. SHOW ALL WORK! No work, no credit.

1. (30 pts) Circle True or False to indicate the validity of the following statements. You do not need to justify your answers. In all of the following, assume that $X_1, ..., X_n$ is a SRS from some population of interest with mean μ and variance σ^2.

   (a) True or False: $Var(X_1 - X_2) = Var(X_1) - Var(X_2)$ when the rvs X_i are independent.

   (b) True or False: Assuming normal data with a range of 8, to construct a 95% CI for μ with a margin of error $m = 1$ you should collect a sample of size $n = 16$.

   (c) True or False: The sum of independent normal rvs has a normal distribution.

   (d) True or False: The t-distribution and the standard normal distribution are both symmetric, bell-shaped, and centered at 0.

   (e) True or False: If $\hat{p}$ is a sample proportion, then $\lim_{n\to\infty} Var(\hat{p}) = 0$.

   (f) True or False: A CI for a population mean μ will become more narrow as the confidence level is increased.

   (g) True or False: Bin(n,$p = .5$) has larger variance compared to any other binomial pmf.

   (h) True or False: You measure the light intensity on $n = 3$ randomly chosen days in January, 2017 in Bozeman. Assuming that σ is known, a 99% CI for the true mean light intensity in January 2017 is $\bar{X} \pm 2.576\frac{\sigma}{\sqrt{3}}$.

   (i) True or False: A 90% confidence interval for $\mu_1 - \mu_2$ is [-10, -1/2]. This CI suggests that $\mu_2 > \mu_1$.

   (j) True or False: $X_1/2$ has smaller variance than $\frac{X_1+X_2}{2}$ when X_1 and X_2 are independent.

   (k) True or False: $X_1/2$ and $\frac{X_1+X_2}{2}$ are both unbiased estimators of the population mean μ.

   (l) True or False: If Bias($\hat{\theta}$) = 2 and Var($\hat{\theta}$) = 3, then MSE($\hat{\theta}$) = 5.

   (m) True or False: You should use a pooled two-sample t-test when you are not sure if both populations have the same variance.

   (n) True or False: $\frac{1}{n}\sum(X_i - \bar{X})^2$ is an unbiased estimator of σ^2.

   (o) True or False: A lower one-sided CI for σ_1^2/σ_2^2 is $[S_1^2/S_2^2 F_{1-\alpha/2}, \infty)$.

   (p) True or False: If $\hat{\theta}$ is a consistent estimator of θ then $\lim_{n\to\infty} \hat{\theta} = \theta$.

2. (9 pts) List the THREE desirable properties that we have discussed in class of a point estimator $\hat{\theta}$ for some parameter θ.

   1. unbiased
   2. small or minimum variance
   3. consistency

3. Suppose that $X_1, X_2, ..., X_n$ are a SRS from $EXP(\beta)$.

(a) (12 pts) Derive the distribution of $U = \frac{\sum X_i}{\beta}$.

$$m_X(t) = (1-\beta t)^{-1}$$

By independence of X_i's,

$$m_{\Sigma X_i}(t) = m_{X_1}(t) \times m_{X_2}(t) \times \cdots \times m_{X_n}(t).$$

Because X_i's are identically $EXP(\beta)$

$$m_{\Sigma X_i}(t) = (1-\beta t)^{-1} \times (1-\beta t)^{-1} \times \cdots \times (1-\beta t)^{-1}$$

$$= (1-\beta t)^{-n}$$

Because $m_{cX}(t) = m_X(ct)$ for any constant c,

$$m_{\Sigma X_i/\beta}(t) = \left(1 - \beta \frac{t}{\beta}\right)^{-n} = (1-t)^{-n}$$

this is the mgf for $\boxed{GAM(n, 1) \sim U.}$

(b) (8 pts) Show that $U = \frac{\sum X_i}{\beta}$ is a pivotal quantity for $E(X_i) = \beta$.

1. U is function of data (via ΣX_i) and of β.

2. U has a dist that is independent of β. (By #3a)

$\therefore$ U is a pivotal quantity.

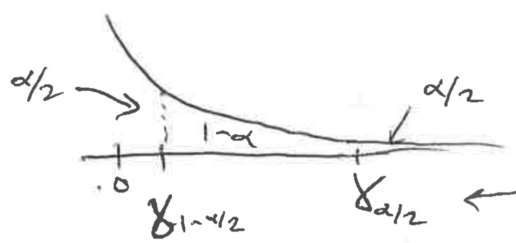
(c) (12 pts) Use the pivotal quantity $U = \frac{\sum X_i}{\beta}$ to construct a two-sided $100(1-\alpha)\%$ CI for $E(X_i) = \beta$. Clearly define each factor in the confidence limits.

By #3a,

$$P\left(\gamma_{1-\alpha/2} \le \frac{\Sigma X_i}{\beta} \le \gamma_{\alpha/2}\right) = 1-\alpha$$

$$P\left(\frac{\Sigma X_i}{\gamma_{\alpha/2}} \le \beta \le \frac{\Sigma X_i}{\gamma_{1-\alpha/2}}\right) = 1-\alpha.$$

$\therefore$ $\left[\Sigma X_i / \gamma_{\alpha/2},\ \Sigma X_i / \gamma_{1-\alpha/2}\right]$ is a $100 \times (1-\alpha)\%$ CI for β.



Book's notation.

(d) (6 pts) Starting with the result in #3c, generate a two-sided $100(1-\alpha)\%$ confidence interval for $\text{Var}(X_i)$. Justify each step in your derivation.

Because $P\left(\frac{\Sigma X_i}{\gamma_{\alpha/2}} \le \beta \le \frac{\Sigma X_i}{\gamma_{1-\alpha/2}}\right) = 1-\alpha$

then $P\left(\left(\frac{\Sigma X_i}{\gamma_{\alpha/2}}\right)^2 \le \beta^2 \le \left(\frac{\Sigma X_i}{\gamma_{1-\alpha/2}}\right)^2\right) = 1-\alpha$

when $\text{Var}(X_i) = \beta^2$.

$\therefore$ $\left[\left(\frac{\Sigma X_i}{\gamma_{\alpha/2}}\right)^2, \left[\frac{\Sigma X_i}{\gamma_{1-\alpha/2}}\right]^2\right]$

is a $100 \times (1-\alpha)\%$ CI for

$\text{Var}(X_i) = \beta^2$.

4. (8 pts) State the Central Limit Theorem. List all assumptions, and give a proper conclusion.

Let $X_1, \dots, X_n$ be iid with $EX_i = \mu$ and $Var(X_i) = \sigma^2 < \infty$.

Then $\lim_{n \to \infty} \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = U \sim N(0,1)$.

5. (12 pts) Let $T \sim t(\nu)$. prove that $T^2 \sim F(1, \nu)$. Justify each step of your proof.

By definition,

$T = \frac{Z}{\sqrt{W/\nu}}$ where $Z \sim N(0,1)$ and $W \sim \chi^2(\nu)$.

So $T^2 = \frac{Z^2}{W/\nu}$. Because $Z \sim N(0,1)$ $U = Z^2 \sim \chi^2(1)$.

$\therefore T^2 = \frac{U/1}{W/\nu}$ = ratio of independent χ^2 rvs.

$= F \sim F(1, \nu)$ by definition.