KEY

Exam 2

STAT422 *April 1, 2011*

Instructions: This exam is worth 100 points. SHOW ALL WORK! No work, no credit.

1. (22 pts) Circle True or False to indicate the validity of the following statements. You do not need to justify your answer. In all of the following, assume that $X_1, ..., X_n$ is a SRS from some population of interest with mean μ and variance σ^2.

   (a) **T** / **F** You should use a pooled two-sample t test when both populations of interest have the same variance.

   (b) **T** / **F** If $\hat{\theta}$ is an MLE, then $\hat{\theta}$ is an MVUE.

   (c) **T** / **F** If $\hat{\theta}$ is an MLE, then $\hat{\theta}$ has an approximate normal distribution for large n only when $\hat{\theta}$ is unbiased for θ.

   (d) **T** / **F** The posterior for a population mean μ, when calculated using a normal likelihood, a non-informative prior, and known population variance, gives a 95% credible interval for μ which is equal to the a 95% confidence interval for μ

   (e) **T** / **F** The following is a valid pair of statistical hypotheses: $H_0: p > 0.6$ versus $H_a: p = 0.7$.

   (f) **T** / **F** The following is a valid pair of statistical hypotheses: $H_0: \overline{X} = 4$ versus $H_a: \overline{X} > 4$.

   (g) **T** / **F** Rejecting H_0 implies H_a is true.

   (h) **T** / **F** A Type II Error is made if a false null hypothesis is rejected.

   (i) **T** / **F** $\overline{X}_1 - \overline{X}_2$ is an unbiased estimator of the population mean $\mu_1 - \mu_2$.

   (j) **T** / **F** Bayesian MAP estimators are always biased. — e.g. $\hat{\mu}_{MAP} = \overline{X}$ for normal likelihood and uninformative $p(\mu)$.

   (k) **T** / **F** The larger the p-value, the stronger the evidence that H_0 is true.

   (l) **T** / **F** The Satterthwaite degree of freedom calculation used in a Welch 2-sample t-test is always between $\min(n_1 - 1, n_2 - 1)$ and $n_1 + n_2 - 2$.

   (m) **T** / **F** If an investigator fails to reject H_0, then the data provide strong support for H_0.

2. Consider statements I, II, III, and IV. Which pair of statements is **TRUE**? Given the sample standard deviation s remains constant, the confidence interval for μ will become

   A. I. and III.
   B. II. and IV.
   C. I. and IV.
   D. II. and III.

   I. Wider as the sample size increases.
   II. Narrower as the sample size increases.
   III. Wider as the confidence level increases.
   IV. Narrower as the confidence level increases.

3. (3 pts) For which of the following scenarios does the probability of a Type II error decrease? CIRCLE ALL THAT APPLY.

   A. increase sample size
   B. decrease sample size
   C. increase α
   D. decrease α
   E. increase the margin of error (the distance between the null and alternative values)
   F. decrease the margin of error (the distance between the null and alternative values)

4. A Type I error is committed if ____________

   A. H_0 is rejected when H_0 is true
   B. H_0 is rejected when H_0 is false
   C. H_0 is not rejected when H_0 is true
   D. H_0 is not rejected when H_0 is false

5. A Type II error is committed if ______________

   A. H_0 is rejected when H_0 is true

   B. H_0 is rejected when H_0 is false

   C. H_0 is not rejected when H_0 is true

   D. H_0 is not rejected when H_0 is false

6. When testing $H_o : p_1 = p_2$ against the alternative, $H_a : p_1 < p_2$, the quantity $Z = \frac{\hat{p}_1 - \hat{p}_2}{SE_{(\hat{p}_1 - \hat{p}_2)}}$ is a random variable having a N(0, 1) distribution (approximately)

   A. when H_a is true and samples are large.

   B. when H_0 is true and samples are large.

   C. when samples are large. It makes no difference whether $p_1 = p_2$ or $p_1 < p_2$.

7. FILL IN THE THREE BLANKS:
   Suppose that $\hat{\theta}_n$ estimates θ for any sample size n. If $Bias(\hat{\theta}_n) \to 0$ and $Var(\hat{\theta}_n) \to 0$, then $\hat{\theta}_n$ is converges in probability to θ. In other words, $\hat{\theta}_n$ is consistent for θ.

   General Weak Laws of Large Numbers.

8. Under what conditions is the MAP equal to the Bayesian mean of the posterior?

   When posterior is unimodal and symmetric.

Well! New in 2011!!

9. Twenty one law enforcement officers agreed to participate in a study of how the newfangled "Tasers" affect the outcome of encounters with combative suspects. Eleven officers were randomly assigned to each of two groups. The officers in first group carried Tasers, and used the Tasers to subdue combative suspects. The officers in the second group did not carry Tasers, and used traditional grappling and wrestling techniques to subdue combative suspects. The number of injuries per encounter was recorded.

   (a) Using a t-distribution, a 95% CI for $\mu_1 - \mu_2$ is $[-4.5, -.4]$. Using this CI, indicate clearly what decision you would make regarding the hypotheses $H_0 : \mu_1 = \mu_2$ versus $H_a : \mu_1 < \mu_2$.

   Because the CI is below 0, we would REJECT Ho in favor of Ha.

   (b) Give a conclusion in terms of the problem.

   We are 95% confident that the mean number of injuries per encounter is between 0.4 and 4.5 more when grappling and wrestling suspects than when using a Taser.

   (c) Are the assumptions required for the test satisfied? Explain.

   there are only 5 officers in 1 group, and 6 in the other. these are small sample sizes. We would have to assume that the number of injuries per encounter are approximately normal in order for the CI to be valid. Also, we need to ask whether the officers in the study are a RS of all officers.

10. Let y_i have pdf $f_y = \frac{\theta-1}{y^\theta}$ for $y \geq 1$.

(a) Assuming that a SRS $y_1, ..., y_n$ is collected, give the lieklihood.

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta - 1}{y_i^{\theta}} = (\theta - 1)^n \left(\prod y_i\right)^{-\theta}$$

(b) Assuming a flat non-informative prior for θ, find the posterior $p(\theta|y_1, ..., y_n)$.

$p(\theta) \propto 1$ so

$$p(\theta|y) \propto L(\theta) \times p(\theta) = (\theta-1)^n \left(\prod y_i\right)^{-\theta} \times 1$$

(c) Find the MAP estimator of θ. Be sure to show that the value you find is a maximum.

$$\ln p(\theta|y) = n \ln(\theta - 1) - \theta \ln\left(\prod y_i\right)$$

$$\frac{d}{d\theta} \ln p(\theta|y) = \frac{n}{\theta - 1} - \ln\left(\prod y_i\right) = 0$$

$$\Rightarrow \hat{\theta}_{MAP} - 1 = \frac{n}{\ln \prod y_i} \Rightarrow \boxed{\hat{\theta}_{MAP} = \frac{n}{\sum \ln y_i} + 1}$$

(d) Without doing any calculations, comment on whether you expect the MLE to be the same as the MAP, or different.

Yes, ought to be the same because of non-informative prior.

11. A political activist decides to randomly choose TWO Montana citizens and ask "Do you support the proposed travel plan for the Gallatin National Forest?" One says yes, one says no.

(a) Give the MLE for p. $\hat{p} = \frac{1}{2}$.

(b) Give the posterior distribution of p given data.

Likelihood $= p(1-p)$; prior $= 1$

$p(p|y) \propto p(1-p) \propto$ Beta$(2, 2)$.

(c) Give the Bayesian posterior mean estimator for p.

$$\hat{p}_B = E[p|\underset{\sim}{y}] = \frac{\alpha}{\alpha+\beta} = \frac{2}{4} = \frac{1}{2}.$$

(d) Give a 75% credible for p.

$$[\text{qbeta}(0.125, 2, 2), \text{qbeta}(.875, 2, 2)] = [0.22, 0.79]$$

(e) Interpret the credible interval in terms of the problem

With probability 0.75, 22% – 79% of Montanans support the Gallatin travel plan.

12. A study is being conducted to determine whether drivers at or over the age of 70 have longer reaction times in a simple driving task than drivers aged less than 70. A SRS of 10 drivers at or over age 70, and another SRS of 11 drivers under age 70 is selected. Each driver's reaction time to a specific driving activity was measured in a laboratory, and the reaction times in minutes were recorded. The sample mean and sample standard deviation of each sample are given in the following table:

	mean	std
Age 70 or over:	0.7720	0.1020
Age under 70:	0.7140	0.0847

(a) State the null and alternative hypotheses.

$H_0: \mu_{<70} = \mu_{70}$

$H_a: \mu_{<70} < \mu_{70}$

(b) Do we need to assume anything about the distribution of driving times in order to proceed with a hypothesis test? Why or why not?

Because the sample sizes of 10 and 11 are small (less than 30), then in order to assume normality of sample means, we need to assume normality of reaction times for both groups.

(c) Using a pooled variance estimate, give the test statistic.

$$S_p^2 = \frac{9(0.1020)^2 + 10(0.0847)^2}{19}$$

$$= 0.0087$$

(d) Give the distribution of the test statistic assuming that the null is true.

$$t \sim t(19), \quad t = \frac{.7140 - .7720}{\sqrt{0.0087\left(\frac{1}{10} + \frac{1}{11}\right)}} = -1.42$$

(e) Calculate the p-value. Draw the picture of the appropriate pdf and shade the region corresponding to the p-value..

$$\text{p-value} = P(t < -1.42) = 0.086$$

(f) Give the rejection region for this test.

At $\alpha = 0.05$, $RR = \{ t < t_{1-\alpha, df=19} = -1.73 \}$

(g) Make a decision at $\alpha = 0.05$.

Because p-value > 0.05 (or because $t = -1.42 \notin RR$) FTR H_0.

(h) State a conclusion IN TERMS OF THE PROBLEM.

The evidence fails to suggest that drivers 70 and over have longer reaction times, on average, compared to younger drivers.

13. According to the Center for Disease Control (CDC), 76 million people in the US get diarrhea and upset stomachs each year. Most of these infections can be prevented by regularly washing one's hands. A microbiologist believes that a majority (i.e. more than 50%) of women wash their hands after using the bathroom. She collects a sample of 40 women, calculates $\hat{p}$, and performs a hypothesis test at $\alpha = .05$.

(a) (6 pts) Give the hypotheses which the microbiologist wants to test.

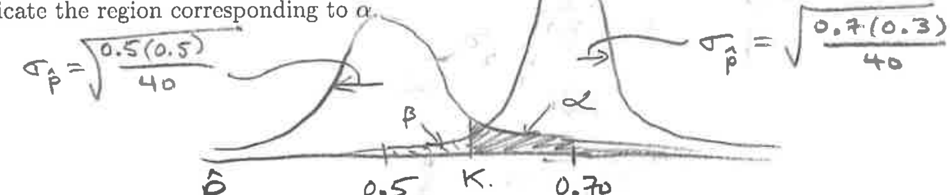
$H_0: p = 0.5$

$H_a: p > 0.5$

(b) (4 pts) Describe a Type II error **in terms of this problem.**

Failing to find that most women wash their hands when in fact most women <u>do</u> wash their hands.

(c) (4 pts) If the H_0 from (13a) is true, sketch the sampling distribution for $\overline{X}$, and shade and clearly indicate the region corresponding to α.



(d) (4 pts) Now suppose that the alternative hypothesis is true, such that the true proportion of women who wash their hands is $p = 0.70$. Add to the graph you made in (13c) a sketch of the sampling distribution for $\overline{X}$, and shade and clearly indicate the region corresponding to β, the probability of a Type II error.

(e) (8 pts) Suppose that the true proportion of women who wash their hands is $\pi = 0.70$ and that H_0 will be rejected if the test statistic $z > z_{.95} = 1.645$ (which corresponds to a test with $\alpha = .05$). For the hypotheses in problem (13a), compute β, the probability of a Type II error.

the $RR = \{ \hat{p} \mid \hat{p} > K \}$ with $k = p_0 + z_\alpha \sqrt{\frac{0.5(0.5)}{40}} = 0.5 + 1.645(0.079) = 0.63$

$\therefore \beta = P(\hat{p} < K) = P(\hat{p} < 0.63) = P\left(z < \frac{0.63 - 0.7}{\sqrt{\frac{.7(.3)}{40}}}\right) =$

$= P(z < -0.97) = \boxed{0.167}$

(f) (4 pts) The microbiologist submits a paper presenting the hand-washing study for publication in a peer-reviewed journal. A reviewer complains that the probability of a Type II error is too large. If the microbiologist were to run a new study in order to be sure that the Type II error probability is no more than 0.05, how many people must she sample?

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{m^2} = \frac{(1.645 + 1.645)^2 (0.5(0.5))}{(0.5 - 0.7)^2} \leftarrow$$

Used $p = 1/2$ to maximize $\sigma^2 = p(1-p)$

$$= 67.7 \longrightarrow \boxed{68}$$

14. In June 2006, researchers at Wake Forest University in North Carolina concluded that long term diets containing trans-fats led to alarming patterns of weight gain and insulin resistance in monkeys (*Discover*, December 2006). Two groups of vervet monkeys were randomly assigned to one of two different diets: a common "western style" trans-fat diet or a "mono-saturated fat" diet (like the fats present in olive oil). Over six years, suppose that the twenty-two monkeys in the "mono-fat" group had the following weight gain statistics (in percentages), $\bar{y} = 1.8$ and $\sigma = .9$.

(a) Previous research suggested that a mean of 3 and a standard deviation of 20. Assuming normality of the data, use a conjugate prior to give a posterior for μ

$$p(\mu) = N(3, 20^2). \quad (\text{ie: } \eta = 3,\ \delta^2 = 20^2).$$

$$\therefore\ p(\mu | \underset{\sim}{y}) = N\left(\frac{\frac{3}{20^2} + \frac{22}{0.9^2}(1.8)}{\frac{1}{20^2} + \frac{22}{0.9^2}},\ \frac{1}{\frac{1}{20^2} + \frac{22}{(0.9)^2}} \right)$$

$$= N(1.8,\ 0.037)$$

(b) Give a credible interval for μ

$$1.8 \pm 1.96\sqrt{0.037} = [1.42,\ 2.18]$$

(c) Interpret in terms of the problem.

With probability 0.95, the true weight gain of monkeys on the trans-fat diet is between 1.42 and 2.18 pounds, on average.