KEY

Exam 2

STAT422 *April 3, 2017*

This exam is worth 100 points. SHOW ALL WORK! No work, no credit.

1. (20 pts) Circle True or False to indicate the validity of the following statements. Assume that $X_1, ..., X_n$ is a SRS from some population with mean μ, variance σ^2, and a likelihood that satisfies any required regularity conditions.

   (a) **T** / **F** $Var(X_1 - \log(X_2)) = Var(X_1) + Var(\log(X_2))$ when the data X_i are independent.

   (b) **T** / **F** For the normal distribution $N(\mu, \sigma^2)$ when σ^2 is unknown, $\bar{X}$ is the MVUE for μ.

   (c) **T** / **F** For Bernoulli data with true proportion of "successes" p, $\hat{p}$ is the MVUE for p.

   (d) **T** / **F** If $\hat{p}$ is the MLE for p, then $\frac{\hat{p}(1-\hat{p}}{n}$ is consistent for $\frac{p(1-p)}{p}$.

   (e) **T** / **F** For any SRS $X_1, ..., X_n$ from any distribution, the MLE for σ is $\frac{1}{n}\sum_i (X_i - \bar{X})^2$.

   (f) **T** / **F** The MOM estimator for $E(X^3)$ is $\frac{1}{n}\left(\sum_{i=1}^n X_i\right)^3$.

   (g) **T** / **F** If S^2 is the sample variance, then $\lim_{n\to\infty} Var(S^2) = 0$.

   (h) **T** / **F** An MLE is always a function of a sufficient statistic.

   (i) **T** / **F** When the data are Bernoulli and a non-informative flat prior is used for p, then the 95% credible interval for p is the same as the 95% confidence interval for p.

   (j) **T** / **F** When the data are normal $N(\mu, \sigma^2)$, σ^2 is known, and a non-informative flat prior is used for μ, then the 95% credible interval for μ is the same as the 95% confidence interval for μ.

2. (12 pts) FILL IN THE BLANKS: Let $\hat{\theta}_{MLE}$ be the MLE for a parameter θ. Assume that the likelihood satisfies any required regularity conditions.

   (a) $(\hat{\theta}_{MLE} - \theta)/\sqrt{\text{Var}(\hat{\theta}_{MLE})} \sim$ $N(0,1)$ for large sample sizes.

   (b) For large sample sizes, $\hat{\theta}_{MLE}$ has MINIMUM variance, and the variance is approximately equal to Cramer-Rao lower bound.

   (c) $\hat{\theta}_{MLE}$ converges in probability to θ.

3. (6 pts) If the MLE is unbiased, explain why the MLE is "typically" equal to the MVUE. Cite the name of the relevant theorem.

By Rao-Blackwell theorem, the MVUE is a function of a sufficient statistic. Because an MLE is always a function of a sufficient statistic, then unbiased MLES are "typically" the MVUE.

4. Consider a SRS $Y_1, ..., Y_n$ with pdf $f(y) = \theta e^{-y\theta}$ for $\theta > 0$ and $y > 0$.

(a) (6 pts) Derive the likelihood function. Where in your derivation are you using independence?

$L(\theta) = \prod_{i=1}^{n} \theta e^{-y_i\theta}$; likelihood is product of pdfs because of independence of data

$= \theta^n e^{-\theta \sum y_i}$

(b) (8 pts) Use the factorization theorem to find a sufficient statistic for θ. Show your work.

$$L(\theta) = \underbrace{\theta^n e^{-\theta \sum y_i}}_{g(\theta, \sum y_i)} \times \underbrace{1}_{h(\underline{y})}$$

By Factorization theorem, $\sum y_i$ is sufficient for θ.

(c) (8 pts) Find the MLE for θ. Show your work. In the interest of time, you do NOT need to perform the 2nd derivative test.

$\ln L(\theta) = n \ln\theta - \theta \sum y_i$

$\Rightarrow \frac{d}{d\theta} \ln L(\theta) = \frac{n}{\theta} - \sum y_i = 0.$

$\Rightarrow \hat{\theta}_{MLE} = \frac{n}{\sum y_i} = 1/\bar{Y}.$

(d) (6 pts) Find the MLE for $\text{Var}(Y) = 1/\theta^2$. State which property of MLEs you are using.

By Invariance Property of MLES, $1/(\hat{\theta}_{MLE})^2 = \frac{1}{(1/\bar{y})^2}$

$= \bar{Y}^2$

is MLE of $1/\theta^2$.

(e) (8 pts) Assuming a flat non-informative prior for θ, find the posterior $p(\theta|y_1, ..., y_n)$. Show your work. This posterior is of a known distribution type. Give the distribution, and specify the parameters.

$$p(\theta|\underset{\sim}{y}) \propto p(\underset{\sim}{y}|\theta) \times p(\theta)$$

$$= \theta^n e^{-\theta \sum y_i} \times 1$$

$$= \theta^{(n+1)-1} e^{-\theta/(1/\sum y_i)}$$

$$\propto \text{GAM}\left(\alpha^* = n+1,\ \beta^* = \frac{1}{\sum y_i}\right).$$

by back of book.

(f) (6 pts) Give the Bayesian mean of the posterior, $\hat{\theta}_B$.

$$\hat{\theta}_B = E(\theta|\underset{\sim}{y}) = \boxed{\alpha^* \beta^* = (n+1)/\sum y_i}$$

(g) (6 pts) Without doing any calculations, explain whether you expect the MAP estimator for θ to be the same as the MLE you found in #4c, or different.

The MLE will be the same as the MAP because a flat non-informative prior was used for θ.

5. You are given iid data $Y_1, ..., Y_n$ with mean μ and variance σ^2.

(a) (6 pts) Find $Var(\bar{Y})$. Indicate the step where you use independence of the data.

$$Var(\bar{Y}) = Var\left(\frac{1}{n}\sum Y_i\right) = \frac{1}{n^2} Var\left(\sum Y_i\right)$$

$$= \frac{1}{n^2}\sum Var(Y_i) \quad \text{by independence}$$

$$= \frac{1}{n^2} n\sigma^2 \quad \text{because identically dist.}$$

$$= \sigma^2/n.$$

(b) (8 pts) Use the General Weak Law of Large Numbers to prove that $\bar{Y}$ is consistent for μ.

$Bias(\bar{Y}) = 0$ because $E\bar{Y} = E\left(\frac{1}{n}\sum Y_i\right) = \frac{1}{n}E\sum Y_i$

$= \frac{1}{n}\sum EY_i = \frac{1}{n}n\mu = \mu.$

$$\lim_{n\to\infty} Var(\bar{Y}) = \lim_{n\to\infty}\frac{\sigma^2}{n} = 0.$$

$\therefore$ By General WLLN, $\bar{Y} \xrightarrow{p} \mu.$

6. EXTRA CREDIT: Show that the MLE that you found in #4d is biased but consistent for $\frac{1}{\theta^2}$.

$\widehat{1/\theta^2} = \bar{Y}^2$. $E(\bar{Y}^2) = Var(\bar{Y}) + E(\bar{Y})^2 = \frac{\sigma^2}{n} + \mu^2$

$= \frac{1}{\theta^2 n} + \mu^2$. $\mu = \frac{1}{\theta}$ because $f(y) = EXP(\beta = \frac{1}{\theta})$

Or derivation from scratch: $\mu = EY = \int_0^\infty y\theta e^{-\theta y}dy$

$= \theta y e^{-\theta y}\left(-\frac{1}{\theta}\right)\Big|_0^\infty - \int_0^\infty \theta e^{-\theta y}\left(-\frac{1}{\theta}\right)dy$ by integration by parts

$= \int_0^\infty e^{-\theta y}dy = -e^{-\theta}\frac{1}{\theta}\Big|_0^\infty = \frac{1}{\theta}.$

$\therefore E(\bar{Y}^2) = \frac{1}{\theta^2 n} + \frac{1}{\theta^2} = \frac{n+1}{n\theta^2}$ so $\bar{Y}^2$ is biased for $\frac{1}{\theta^2}$

but $\lim_{n\to\infty} E\bar{Y}^2 = \lim_{n\to\infty}\frac{n+1}{n\theta^2} = \frac{1}{\theta^2}$. Needed by WLLN

For large n, $\mathrm{Var}(\overline{Y}^2) \to$ CRLB. Numerator is

$$\left(\frac{d}{d\theta} t(\theta)^2\right) = \left(\frac{d}{d\theta}\frac{1}{\theta^2}\right)^2 = \left(\frac{-2}{\theta^3}\right)^2 = \frac{4}{\theta^6}$$

Den is $-nE\left[\frac{d^2}{d\theta^2}\ln \theta e^{-y\theta}\right] = -nE\left(\frac{d^2}{d\theta^2}[\ln\theta - y\theta]\right)$

$$= -nE\left[\frac{d}{d\theta}\left[\frac{1}{\theta} - y\right]\right] = -nE\left[-\frac{1}{\theta^2}\right] = \frac{n}{\theta^2}.$$

$$\therefore \text{CRLB} = \frac{4/\theta^6}{n/\theta^2} = \frac{4}{n\theta^4}.$$

$\Rightarrow \mathrm{Var}(\overline{Y}) \longrightarrow 0$ as $n \longrightarrow \infty$

$\Rightarrow \overline{Y}^2 \xrightarrow{P} 1/\theta^2$. by GWLLN