KEY

100 pts

Exam 3

STAT422 *May 2, 2011*

Instructions: This exam is worth 100 points. SHOW ALL WORK! No work, no credit.

30
2 each

1. Circle True or False to indicate the validity of the following statements. You do not need to justify your answer for this problem. In all of the following, assume that $y_1, ..., y_n$ is a SRS from some population of interest with population mean μ and population variance σ^2.
   (a) **T** / **F** For a normal distribution, the MVUE for μ is $\bar{X}$.
   (b) **T** / **F** For a normal distribution, the MLE for σ^2 is $1/n \sum(y_i - \bar{y})^2$.
   (c) **T** / **F** $\sqrt{\frac{1}{n-1}\sum(y_i - \bar{y})^2}$ is an unbiased estimator of σ.
   (d) **T** / **F** If $\hat{\theta}$ is the MLE for θ, then $\hat{\theta}$ is consistent for θ.
   (e) **T** / **F** The MVUE for a parameter θ has the smallest variance when compared to any other point estimator of θ.
   (f) **T** / **F** You should use an un-pooled two-sample t-test when you are not sure if both populations have the same variance.
   (g) **T** / **F** If an investigator fails to reject H_0, then the data provide strong support for H_0.
   (h) **T** / **F** If (.5, 4.5) is a 95% confidence interval for σ^2, then the sample variance is $S^2 = 2.5$.
   (i) **T** / **F** The p-value for the test of two variances is two times the area to the right of the calculated F statistic under the appropriate F distribution curve.
   (j) **T** / **F** Simulations suggest that for sample sizes of non-binary quantitative data larger than 30, the F-test of two variances is robust to deviations of the data from normality.
   (k) **T** / **F** Simulations suggest that for sample sizes of non-binary quantitative data larger than 30, the t-test of two means is robust to deviations of the data from normality.
   (l) **T** / **F** If a 95% confidence interval for $\mu_1 - \mu_2$ is (-43, -21), then μ_2 is larger than μ_1 by between 21 and 43 with 95% confidence.
   (m) **T** / **F** When conducting a Bayesian test of the hypotheses: $H_0 : \mu = \mu_0$ versus $H_a : \mu < \mu_0$, reject H_0 if the posterior probability $p(\mu < \mu_0) < 0.5$.
   (n) **T** / **F** The test statistic for the sign test is a binomial rv.
   (o) **T** / **F** The Mann Whitney test is a non-parametric test of two population means.

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1.5 ea /
.5 blank

2. FILL IN THE BLANKS:
   (a) Recall that if $X \sim \text{Expon}(\theta)$, then $\bar{X}$ is the MLE for $E(X)$. Thus, $\bar{X}^2$ is the MLE for $Var(X)$ by the Invariance Property of MLEs.
   (b) Given a value of δ, when testing whether two variances σ_1^2 and σ_2^2 are equivalent, then the parameter of interest is σ_1^2/σ_2^2 and the equivalence zone is $(\frac{1}{\delta}, \delta]$.
   (c) If a 95% confidence interval for $\mu_1 - \mu_2$ is (-43, -21), then μ_1 is equivalent to μ_2 with 97.5 % confidence as long as differences up to 43 are considered negligible.
   (d) To test the hypotheses $H_0 : \eta = 3$ versus $H_a : \eta \neq 3$ using the Wilcoxon signed rank test, the following $n = 5$ data were collected: $2, 2, \cancel{3}, 4, 6$. The test statistics are $T^- =$ 3 and $T^+ =$ 7.
   1 2 3 4

$\boxed{42}$

2 (e) When conducting 20 tests and using Bonferroni's method to maintain a family-wise Type I error rate of 0.05, the p-value for each individual test must be compared to $\frac{.05}{20} = \frac{.005}{2}$ $= .0025$.

2 3. CIRCLE ONE: Power is the probability that

- **A.** H_0 is rejected when H_0 is true
- **B.** H_0 is rejected when H_0 is false
- **C.** H_0 is not rejected when H_0 is true
- **D.** H_0 is not rejected when H_0 is false

2 4. Give a brief, non-technical description of the Rao-Blackwell Theorem.

The MVUE is a function of a sufficient statistic.

2 5. When should you use the 1-sample sign test instead of the 1-sample Wilcoxon signed rank test?

When the data is NOT from a symmetric distribution or if the data can not be ranked.

2 6. What is the difference between the most powerful test found by the Neyman-Pearson Lemma, and a uniformly most powerful test?

The most powerful test found by Neyman Pearson is for a set of simple hypotheses $H_0: \theta = \theta_0$
$H_a: \theta = \theta_a$.

The UMP is the same as the most powerful test for a one-sided alternative when the RR does not depend on θ_a.

7. For the following theorem statements, clearly state the hypotheses and conclusions:

4 (a) State the Central Limit Theorem.

If $Y_1, \ldots, Y_n$ is a SRS with mean μ and variance $\sigma^2 < \infty$

then $\frac{\bar{Y} - \mu_Y}{S_Y/\sqrt{n}} \xrightarrow{d} N(0,1)$.

$\boxed{14}$

4 (b) State the Weak Law of Large Numbers.

If $y_1, \ldots, y_n$ is a SRS w/ finite variance, then

$$\bar{Y} \xrightarrow{p} \mu_y.$$

4 (c) State the Neyman-Pearson Lemma.

If $y_1, \ldots, y_n$ is a SRS, then the most powerful α-level test of the simple hypotheses $H_0: \theta = \theta_0$, $H_a: \theta = \theta_a$ has $RR = \left\{ \frac{L(\theta_0)}{L(\theta_a)} < k_\alpha \right\}$

4 8. Consider two populations with different means, but the same variance σ^2. A SRS of size n_1 is drawn from population one, which has sample variance S_1^2, an unbiased estimator of σ^2; and independently of the 1st sample, another SRS of size n_2 is drawn from population two, which has sample variance S_2^2, also an unbiased estimator of σ^2. Prove that the 2-sample pooled sample variance $S_p^2 = \frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$ is unbiased.

$$E(S_p^2) = \frac{1}{n_1+n_2-2}\left((n_1-1)E(S_1^2) + (n_2-1)E(S_2^2)\right)$$

$$= \frac{1}{n_1+n_2-2}\left((n_1-1)\sigma^2 + (n_2-1)\sigma^2\right)$$

$$= \sigma^2$$

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9. Let y_i have pdf $f_y = \frac{\theta-1}{y^\theta}$ for $y \geq 1$.

3 (a) Assuming that a SRS $y_1, ..., y_n$ is collected, give the likelihood.

$$L = \prod_i (\theta-1)/y_i^\theta = (\theta-1)^n \prod y_i^{-\theta}$$

4 (b) Give a sufficient statistic for θ. Be clear how you found the statistic, and name the theorem that you are using.

By the Factorization thm, since

$L = g(\prod_i y_i \mid \theta) \underbrace{h(y)}_{1}$, then $\prod_i y_i$ is sufficient for θ.

5 (c) Find the MLE estimator of θ. Be sure to show that the value you find is a maximum.

4. $\ln(L) = n \ln(\theta-1) - \theta \sum \ln y_i$

$$\frac{d}{d\theta} \ln(L) = \frac{n}{\theta-1} - \sum \ln y_i$$

$$\Rightarrow \boxed{\hat{\theta} = \frac{n}{\sum \ln y_i} + 1}$$

1. Using 2nd der test ...

$$\frac{d^2}{d\theta^2} \ln(L) = \frac{-n}{(\theta-1)^2} < 0$$

$\Rightarrow \hat{\theta}$ is the MLE

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2 (d) A Bayesian statistician uses the same likelihood that you give in #9a, then calculates a MAP estimate for θ. Comment on when you would expect the MLE to be the same as the MAP, and when you expect them to be different.

If the prior is flat and non-informative, then MAP = MLE

4 (e) Derive the rejection region for the likelihood ratio test statistic λ for the hypotheses $H_0 : \theta = \theta_0$ versus $H_a : \theta > \theta_0$.

$$\lambda = \frac{\max_{\theta \in \Omega_0} L(\theta)}{\max_{\theta \in \Omega} L(\theta)} = \frac{L(\theta_0)}{L(\hat{\theta}_{MLE})} = \left(\frac{\theta_0 - 1}{\hat{\theta} - 1}\right)^{n} \prod y_i^{\hat{\theta} - \theta_0}$$

$$RR = \left\{ \lambda = \left(\frac{\theta_0 - 1}{\frac{n}{\sum \ln(y)}}\right)^{n} \prod y_i^{\frac{n}{\sum \ln y_i} + 1 - \theta_0} < k_\alpha \right\}$$

$$= \left\{ -2 \ln \lambda > \underbrace{\chi^2_{1-\alpha}(1)}_{\alpha\text{th percentile from } \chi^2(1) \text{ distribution.}} \right\}$$

4 (f) Give the asymptotic distribution of $-2\ln(\lambda)$, and justify the numeric value of the degrees of freedom that you give.

$$-2\ln(\lambda) \,\dot\sim\, \chi^2(r_0 - r) = \chi^2(1)$$

since $r_0 = 1$ parameter, $\theta = \theta_0$, is fixed under H_0, $r = 0$ parameters are fixed under $\Omega_0 \cup \Omega_a$.

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