Solutions Project 2
(20 points)

1. (7.29, 1 pt) By Definition 7.3, a ratio of independent χ^2 rvs, each divided by its degrees of freedom, has an F distribution, $Y = (W_1/\nu_1)/(W_2/\nu_2) \sim F(\nu_1, \nu_2)$. Therefore, by definition, $U = 1/Y = (W_2/\nu_2) \div (W_1/\nu_1)$ has an F distribution with ν_2 numerator and ν_1 denominator degrees of freedom.

2. (7.33, 1 pt) By Definition 7.2, $T = Z/\sqrt{W/\nu} \sim t(\nu)$ if $Z \sim N(0,1)$, $W \sim \chi^2(\nu)$, and if Z and W are independent. Now consider $T^2 = Z^2/(W/\nu)$. Exercise 6.7 showed that $Z^2 \sim \chi^2(1)$. Since Z^2 and W are independent, then $T^2 = (Z^2/1)/(W/\nu)$ is a ratio of independent χ^2 rvs, each divided by their degrees of freedom. This is the definition of an $F(1, \nu)$ distribution.

3. (7.39, 3 pts)

a. Because the data are normal and each sample is independent of the others, then $\overline{X}_i$ have independent a normal distributions with mean μ_i and variance σ/n_i for $i = 1, 2, \ldots, k$. Since $\hat{\theta}$ is a linear combination of independent normal rvs, then Theorem 6.3 shows that $\hat{\theta}$ has a normal distribution with mean given by

$$E(\hat{\theta}) = E(c_1\overline{X}_1 + \ldots + c_k\overline{X}_k) = \sum_{i=1}^{k} c_i\mu_i$$

and variance given by

$$V(\hat{\theta}) = V(c_1\overline{X}_1 + \ldots + c_k\overline{X}_k) = \sigma^2\sum_{i=1}^{k} c_i^2/n\,.$$

b. For $i = 1, 2, \ldots, k$, $(n_i - 1)S_i^2/\sigma^2 \sim \chi^2(n_i - 1)$ (Theorem 7.3). Because the S_i^2 are independent, then

$$\frac{SSE}{\sigma^2} = \sum_{i=1}^{k}(n_i - 1)S_i^2/\sigma^2$$

is a sum of independent chi–square rvs. Thus, $\frac{SSE}{\sigma^2}^2$ has mgf

$\prod(1-2t)^{-(n_i-1)/2} = (1-2t)^{\sum(n_i-1)/2}$ which is the mgf for a $\chi^2(\Sigma(n_i - 1))$ rv. This proves that

$$\frac{SSE}{\sigma^2}^2 \sim \chi^2\left(\sum_{i=1}^{k}(n_i - 1)\right) = \chi^2\left(\sum_{i=1}^{k} n_i - k.\right)$$

c. From part (a), we have that $\dfrac{\hat{\theta}-\theta}{\sigma\sqrt{\sum_{i=1}^{k} c_i^2 / n_i}} \sim N(0,1)$, and in part (b) we showed that $\sum_{i=1}^{k}(n_i-1)S_i^2/\sigma^2 \sim \chi^2\left(\sum_{i=1}^{k} n_i - k.\right)$. Therefore, by Definition 7.2, a rv constructed as

$$\frac{\hat{\theta}-\theta}{\sigma\sqrt{\sum_{i=1}^{k} c_i^2 / n_i}} \Bigg/ \sqrt{\frac{\sum_{i=1}^{k}(n_i-1)S_i^2/\sigma^2}{\sum_{i=1}^{k} n_i - k}} = \frac{\hat{\theta}-\theta}{\sqrt{\mathrm{MSE}\sum_{i=1}^{k} c_i^2 / n_i}},$$

is the ratio of a standard normal rv over the square root of a χ^2 rv divided by its degrees of freedom. By definition, this ratio is distributed as $t\,(\sum_{i=1}^{k} n_i - k)$. Here, we are assuming that $\hat{\theta}$ and SSE are independent (as are $\bar{Y}$ and S^2 in Theorem 7.3).

4. (8.1, 1pt) Let $B = B(\hat{\theta}) = E(\hat{\theta}) - \theta$. Then,

$$MSE(\hat{\theta}) = E\left[(\hat{\theta}-\theta)^2\right] = E[\hat{\theta}-E(\hat{\theta})+E(\hat{\theta})-\theta] = E\left[(\left(\hat{\theta}-E(\hat{\theta})\right)+B)^2\right]$$
$$= E\left[\left(\hat{\theta}-E(\hat{\theta})\right)^2\right] + 2B\times E\left[\hat{\theta}-E(\hat{\theta})\right] + E(B^2)$$

The middle term is zero because $E\left[\hat{\theta}-E(\hat{\theta})\right] = E(\hat{\theta}) - E[E(\hat{\theta})] = E(\hat{\theta}) - E(\hat{\theta}) = 0$. Also, $E(B^2) = B^2$ because B is a constant. This shows that $MSE = V(\hat{\theta}) + B^2$.

5. (8.6, 2 pts)

a. $E(\hat{\theta}_3) = aE(\hat{\theta}_1) + (1-a)E(\hat{\theta}_2) = a\theta + (1-a)\theta = \theta$.

b. $V(\hat{\theta}_3) = a^2V(\hat{\theta}_1) + (1-a)^2V(\hat{\theta}_2) = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$, since it was assumed that $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent. To minimize $V(\hat{\theta}_3)$, we can take the first derivative (with respect to a), $\dfrac{d}{da}V(\hat{\theta}_3) = 2a\sigma_1^2 - 2(1-a)\sigma_2^2$ set it equal to zero, to find

$$a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$$

The second derivative test shows that this is indeed a minimum since $\dfrac{d}{da}V(\hat{\theta}_3) = 2\sigma_1^2 + 2\sigma_2^2 > 0$.

6. (8.8, 2 pts)

a. $E\hat{\theta}_1 = EY_1 = \theta$;

$E\hat{\theta}_2 = \dfrac{Y_1 + Y_2}{2} = 1/2(EY_1 + EY_2) = 1/2(\theta + \theta) = \theta$;

$E\hat{\theta}_3 = \dfrac{Y_1 + 2Y_2}{3} = 1/3(EY_1 + 2EY_2) = 1/3(\theta + 2\theta) = \theta$; and

$E\hat{\theta}_5 = \dfrac{Y_1 + Y_2 + Y_3}{3} = 1/3(EY_1 + EY_2 + EY_3) = 1/3(\theta + \theta + \theta) = \theta$

Thus, these four point estimators are *unbiased*. From Ex. 6.81 it was shown that the first order statistic $\hat{\theta}_4 = Y_{(1)}$ ~ EXP(θ/3), so $E\hat{\theta}_4 = \theta/3$, which shows that this estimator is biased. Because you did not study order statistics in STAT421, this problem was graded as extra credit.

b. Since the Y_i are i.i.d exponentially distributed with parameter θ, then it is straightforward to show that $V(\hat{\theta}_1) = \theta^2$, $V(\hat{\theta}_2) = \theta^2/2$, $V(\hat{\theta}_3) = 5\theta^2/9$, and $V(\hat{\theta}_5) = \theta^2/3$, so the estimator $\hat{\theta}_5$ is unbiased and with the smallest variance.

7. (8.13, 2 pts)

a. For a rv Y with the binomial distribution, $E(Y) = np$ and $V(Y) = np(1-p)$, so $E(Y^2) = np(1-p) + (np)^2$. Thus,

$$\begin{aligned} E\left\{n\left(\tfrac{Y}{n}\right)\left[1-\tfrac{Y}{n}\right]\right\} = E(Y) - \tfrac{1}{n}E(Y^2) &= np - \frac{1}{n}(n^2p^2 + np(1-p)) \\ &= np - np^2 - p(1-p) \\ &= np(1-p) - p(1-p) \\ &= np(1-p)\left(\frac{n-1}{n}\right). \end{aligned}$$

b. The unbiased estimator should have expected value $np(1-p)$. Since $E(n\left(\tfrac{Y}{n}\right)\left[1-\tfrac{Y}{n}\right]) = np(1-p)\left(\dfrac{n-1}{n}\right)$, then adjusting by $n/(n\text{-}1)$ gives the unbiased estimator

$$\hat{\theta} = \left(\tfrac{n}{n-1}\right)n\left(\tfrac{Y}{n}\right)\left[1-\tfrac{Y}{n}\right].$$

8. (8.26, 2pts)

a. The estimate of the true proportion p who think humans should be sent to Mars is $\hat{p} = 0.49$ with an error bound of $1.96\sqrt{\tfrac{.49(.51)}{1093}}$ = .029637, giving [0.4603, 0.5197], a 95% CI for p.

b. The standard error is given by $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$. Observe that $\frac{\hat{p}(1-\hat{p})}{n}$ is a parabola with a vertex and maximum $\hat{p}$ = .5. So, a conservative error bound that could be used for all sample proportions (with n = 1093) is $1.96\sqrt{\frac{.5(.5)}{1093}}$ = .029643.

9. (8.48, 3 pts)

From the back cover of your textbook, the mgf for Y_i is $m_Y(t) = (1-\beta t)^{-2}$.

a. Let $U = 2\sum_{i=1}^{n} Y_i / \beta$. By independence of the rv's Y_i, the mgf for U is

$$m_U(t) = E(e^{tU}) = \left[m_Y(2t/\beta)\right]^n = (1-2t)^{-2n}.$$

Thus, $U \sim \chi^2(4n)$. Since U is a function of the data and β with no other unknowns, and the distribution $\chi^2(4n)$ does not depend on β, U is a pivotal quantity for β.

b. From problem #4a,

$$P\left(\chi^2_{.975} \le 2\sum_{i=1}^{n} Y_i / \beta \le \chi^2_{.025}\right) = .95.$$

So, $\left(\dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{.025}}, \dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{.975}}\right)$ is a 95% CI for β.

c. Using percentiles from $\chi^2(20)$, the 95% CI is $\left(\dfrac{2(5)(5.39)}{34.1696}, \dfrac{2(5)(5.39)}{9.59083}\right)$ = (1.577, 5.620).

10. (8.129, 1 pt)

From Exercise 7.20, $V(S^2) = \frac{2\sigma^4}{n-1}$.

a. Since $S'^2 = \frac{n-1}{n} S^2$ then $V(S'^2) = V(\frac{n-1}{n} S^2) = \left(\frac{n-1}{n}\right)^2 V(S^2) = \frac{2(n-1)\sigma^4}{n^2}$.

b. By part (a), $V(S'^2) = \left(\frac{n-1}{n}\right)^2 V(S^2)$. Because $\frac{n-1}{n} < 1$, then $V(S'^2) < V(S^2)$.

11. (2 pts)

a. $\hat{\theta} = \overline{X} + z_{0.25}\sigma = \overline{X} + .6745\sigma$

b. $Var(\hat{\theta}) = Var(\overline{X}) = \sigma^2 / n$

c. Since $P(\overline{X} - 1.645\sigma/\sqrt{n} \le \mu \le \overline{X} + 1.645\sigma/\sqrt{n}) = .90$ then

$P(\overline{X} - 1.645\sigma/\sqrt{n} + .6745\sigma \le \mu + .6745\sigma \le \overline{X} + 1.645\sigma/\sqrt{n} + .6745\sigma) = .9$.

Thus, $\overline{X} + \sigma(.6745 \pm 1.645/\sqrt{n})$ is a 90% CI for θ, the 75th percentile, when σ is known.