Project 4 Solutions
(19 pts)

1. (3 pts)
a. To show that $\hat{\sigma}^2$ is unbiased for σ^2, note that $Y_1 - Y_2 \sim N(0, 2\sigma^2)$. Standardizing $Y_1 - Y_2$ shows that $Z = \frac{Y_1 - Y_2}{\sqrt{2\sigma^2}} \sim N(0,1)$. Theorem 7.2 shows that $Z^2 = \frac{\hat{\sigma}^2}{\sigma^2} = \frac{(Y_1 - Y_2)^2}{2\sigma^2} \sim \chi^2(1)$. Hence $E\left(Z^2\right) = E\left(\frac{\hat{\sigma}^2}{\sigma^2}\right) = E\left(\frac{(Y_1-Y_2)^2}{2\sigma^2}\right) = 1$ which proves that $E(\hat{\sigma}^2) = \sigma^2$.

b. To determine consistency, use the fact that the $\chi^2(1)$ rv has $V(\frac{(Y_1-Y_2)^2}{2\sigma^2}) = 2$ which implies that $V(\hat{\sigma}^2) = 2\sigma^4$. Note that $V(\hat{\sigma}^2) = 2\sigma^4$ is constant for all n which shows that $\lim_{n\to\infty} V\left(\hat{\sigma}^2\right) \neq 0$. Strictly speaking, this means that we cannot apply the General Weak Law of Large Numbers (Theorem 9.1). In fact, $\hat{\sigma}_2^2$ is not a consistent estimator for σ^2 (or any other value).

2. (9.20, 2 pts) Since $E(Y) = np$ and $V(Y) = np(1 - p)$, we have that $E(Y/n) = p$ and $V(Y/n) = 1/n^2\, V(Y) = p(1 - p)/n$. Thus, Y/n is consistent by the General Weak Law of Large Numbers (Theorem 9.1) since it is unbiased and its variance goes to 0 with n.

3. (9.24, 3 pts)

a. By Theorem 7.2, $\sum_{i=1}^{n} Y_i^2 \sim \chi^2(n)$. We will use this fact in part b to show that $E\left(\sum_{i=1}^{n} Y_i^2\right) = n.$

b. By part (a), $E(W_n) = \frac{1}{n} E\left(\sum_{i=1}^{n} Y_i^2\right) = \frac{1}{n} n = 1$ so W_n is an unbiased estimator of 1. Furthermore $V(W_n) = \frac{1}{n^2} V\left(\sum_{i=1}^{n} Y_i^2\right) = \frac{1}{n^2}(2n) = \frac{2}{n}$, which goes to zero as n goes to infinity. By General Weak Law of Large Numbers (Theorem 9.1), as $n \to \infty$, W_n converges in probability to $E(W_n) = 1$.

4. (9.32, 2pts) $EY = \int_2^\infty y \frac{2}{y^2} dy = \int_2^\infty \frac{2}{y} dy = 2\left(\lim_{c\to\infty} \ln(c) - \ln(2)\right) = \lim_{c\to\infty} \ln(c) = \infty$. In other words, EY is undefined which you could argue does not allow you to apply the Weak Law of Large Numbers and stop there. In your book's statement of the Weak Law of Large Numbers (in Example 9.2), the only assumption they state is that VY must be finite. So another approach to this problem (that you were not expected to do) is to show that VY is infinite (that is, undefined). For the sake of completeness, I will check this. First rewrite $VY = E(Y^2) - (EY)^2$. The first term is $E(Y^2) = \int_2^\infty y^2 \frac{2}{y^2} dy = \int_2^\infty 2dy = \lim_{c\to\infty} 2c = \infty$. Thus $VY = E(Y^2) - (EY)^2 = \lim_{c\to\infty}\left(2c - (\ln(c))^2\right)$ = "$\infty - \infty^2$" which is an indeterminate form, so a little more work is required. I did this by rewriting $VY =$

$\lim_{c\to\infty} 2c\left(1-\frac{(\ln(c))^2}{2c}\right)$ then applying L'Hopital's Rule (differentiate the numerator and denominator) to the second factor: *VY* =

$$\lim_{c\to\infty} 2c\left(1-\frac{(\ln(c))^2}{2c}\right)=\lim_{c\to\infty} 2c\left(1-\frac{2\ln(c)\frac{1}{c}}{2}\right)=\lim_{c\to\infty} 2c\left(1-\frac{\ln(c)}{c}\right).$$ Now apply L'Hopital's Rule again to the second factor $VY = \lim_{c\to\infty} 2c\left(1-\frac{\frac{1}{c}}{1}\right)=\lim_{c\to\infty} 2c\left(1-\frac{1}{c}\right)=\lim_{c\to\infty}(2c-2)=\infty.$ This shows that the Weak Law of Large Numbers does not apply since finite variance is required.

5. (Exercise 9.36, 2pts) Let $X_1, X_2, \ldots, X_n$ be a sequence of Bernoulli trials with success probability *p* with $Y=\sum_{i=1}^{n} X_i$. Thus, by the Central Limit Theorem, $U_n = \frac{\hat{p}_n - p}{\sqrt{\frac{pq}{n}}}$ has a limiting standard normal distribution. Exercise 9.20 showed that $\hat{p}_n$ is consistent for *p*. Since $\hat{p}_n(1-\hat{p}_n)$ is a continuous function of $\hat{p}_n$, Theorem 9.2 shows that $\hat{p}_n(1-\hat{p}_n)$ is consistent for *p(1-p)*, or in other words, $\hat{p}_n(1-\hat{p}_n)$ converges in probability to *p(1-p)*. Define $W_n = \sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{p(1-p)}}$, so again by Theorem 9.2, W_n converges in probability to 1. By Slutsky's Theorem (or the weaker form in Theorem 9.3), $\frac{U_n}{W_n} = \frac{\hat{p}_n - p}{\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}}$ converges in distribution to a standard normal variable.

6. (9.38, 3 pts) For a SRS taken from a normal distribution, the likelihood function is given by

$$L(y\mid\theta=(\mu,\sigma^2))=\frac{1}{(2\pi)^{n/2}\sigma^n}\exp\left[-\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{2\sigma^2}\right]$$
$$=(2\pi)^{-n/2}\sigma^{-n}\exp\left[\frac{-1}{2\sigma^2}\left(\sum_{i=1}^{n}y_i^2-2\mu n\bar{y}+n\mu^2\right)\right].$$

a. When σ^2 is known, $\bar{Y}$ is sufficient for μ by the Factorization Theorem 9.4 since the likelihood $L(y|\mu)$ can be factored into two functions *g x h* with $g(\bar{y},\mu)=\exp\left(\frac{2\mu n\bar{y}-n\mu^2}{2\sigma^2}\right)$ and $h(\boldsymbol{y}) = (2\pi)^{-n/2}\sigma^{-n}\exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{n}y_i^2\right)$.

b. When μ is known, $\sum_{i=1}^{n}(y_i - \mu)^2$ is sufficient for σ^2 by Factorization Theorem 9.4 since the likelihood $L(y|\sigma^2)$ can be factored into two functions *g x h* with

$$g\left(\sum_{i=1}^{n}(y_i - \mu)^2, \sigma^2\right) = (\sigma^2)^{-n/2} \exp\left[-\frac{\sum_{i=1}^{n}(y_i - \mu)^2}{2\sigma^2}\right] \text{ and } h(\mathbf{y}) = (2\pi)^{-n/2}.$$

c. See Example 9.8.

7. (9.42, 1pt) The likelihood function is $L(\mathrm{y}|p) = \prod_{i=1}^{n} p(1-p)^{y_i - 1} = p^n(1-p)^{\Sigma y_i - n} = p^n(1-p)^{n\bar{y}-n}$. By Factorization Theorem 9.4, $\overline{Y}$ is sufficient for *p* since the likelihood $L(\mathrm{y}|p)$ factors into two functions *g x h* with $g(\bar{y}, p) = p^n(1-p)^{n\bar{y}-n}$ and $h(\mathbf{y}) = 1$.

8. (9.56, 2 pts)) In Exercise 9.38b, it was shown that $\sum_{i=1}^{n}(y_i - \mu)^2$ is sufficient for σ^2. Since $\sum_{i=1}^{n}\frac{(y_i - \mu)^2}{\sigma^2} \sim \chi^2(n)$, then $E\left(\sum_{i=1}^{n}\frac{(y_i - \mu)^2}{\sigma^2}\right) = n$ which shows that $E\left(\sum_{i=1}^{n}(y_i - \mu)^2\right) = n\sigma^2$ and so $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mu)^2$ is unbiased for σ^2. By the Rao Blackwell Theorem 9.5, since the quantity $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mu)^2$ is unbiased and a function of the sufficient statistic, it is the MVUE of σ^2. Note that $1/n$ is the multiplier out front and NOT $1/(n-1)$ since μ is known.

9. (1 pt) Problem #6 in this HW showed that $\overline{Y}$ sufficient for *p*. Because $1/p$ is an invertible function of *p*, $\overline{Y}$ is also sufficient for $1/p$. Since $E(\overline{Y}) = E(Y)$, and $E(Y) = 1/p$ when $Y \sim \mathrm{GEO}(p)$, then we have that $\overline{Y}$ is unbiased and sufficient for $1/p$. Rao-Blackwell now shows that $\overline{Y}$ is the MVUE for $1/p$.