Solutions Project 5
15 points for undergrads, 16 points for grads

1. (Exercise 9.70, 1pt) MOM estimates the first population moment EY with the first sample moment $\bar{Y}$. Since $EY = \lambda$ for a POI(λ) rv, the MOM estimator of λ is $\hat{\lambda} = \bar{Y}$.

2. (Exercise 9.74, 2pts)

a. First, calculate $EY = \int_0^\theta 2y(\theta - y)/\theta^2 dy$ = $\theta/3$. Because MOM estimates the first population moment $EY = \theta/3$ with the first sample moment $\bar{Y}$, then the MOM estimator of θ is $\hat{\theta} = 3\bar{Y}$.

b. The likelihood is $L(\theta) = 2^n \theta^{-2n} \prod_{i=1}^n (\theta - y_i)$. Therefore the likelihood can't be factored into a function that only depends on $\bar{Y}$ and θ because, e.g., $\prod_{i=1}^n y_i / \theta^2$ is a term in the likelihood. Therefore, the MOM estimator is not a sufficient statistic for θ.

3. (Exercise 9.80, 3pts)

a. In class we found that the MLE for λ was $\hat{\lambda} = \bar{Y}$.
b. Because $E\bar{Y} = EY = \lambda$ then $E(\hat{\lambda}) = \lambda$. Because $V\bar{Y} = VY/n$ and $VY = \lambda$ then $V(\hat{\lambda}) = \lambda/n$.
c. Since $\hat{\lambda}$ is unbiased and has a variance $V(\hat{\lambda}) = \lambda/n$ that goes to 0 as n goes to infinity, then $\hat{\lambda}$ is consistent for λ.
d. By the invariance property of MLEs, the MLE for $P(Y = 0)=\exp(-\lambda)$ is $\exp(-\hat{\lambda})$.

4. (2 pts) For iid EXP(θ) rvs, the likelihood is $L(\theta) = f(y_1, \dots, y_n|\theta) = \frac{1}{\theta^n} e^{\sum y_i/\theta}$. The log likelihood is $\ln \mathrm{L}(\theta) = -n \ln \theta - \sum y_i/\theta$ so $\frac{d}{d\theta} \ln L(\theta) = -\frac{n}{\theta} + \sum y_i/\theta^2$. Setting the derivative to zero shows that the MLE $\hat{\theta}$ satisfies $-n + \sum \frac{y_i}{\hat{\theta}} = 0$, so $\hat{\theta} = \bar{Y}$ is a critical point. The second derivative of the log likelihood is $\frac{d^2}{d\theta^2} \ln L(\theta) = \frac{n}{\theta^2} - 2\sum y_i/\theta^3$. Evaluating at the critical point, $\frac{d^2}{d\theta^2} \ln L(\theta = \bar{Y}) = \frac{n}{\bar{Y}^2} - 2\sum \frac{y_i}{\bar{Y}^3} = \frac{-n}{\bar{Y}^2} < 0$. By the second derivative test $\hat{\theta} = \bar{Y}$ is the MLE for θ.

5. (Exercise 9.81, 1pt) By #4, the MLE is $\hat{\theta} = \bar{Y}$. By the invariance property of MLEs, the MLE of $VY = \theta^2$ is $\bar{Y}^2$.

6. (2 pts) See Example 9.15 on page 478 of your textbook, which shows that $\hat{\mu} = \bar{Y}$ is a critical point. Your book omits the second derivative test, which is easy: $\frac{d^2}{d\mu^2} \log(L(\mu)) = -n/\sigma^2$ is always negative. That is, $L(\mu)$ is a concave function. Thus, $\hat{\mu} = \bar{Y}$ is the MLE.

7. (Exercise 9.96, 2 pts) From Example 9.15, the MLE for σ^2 was found to be $(S')^2 = \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$.

(a) By the invariance property of MLEs, the MLE for $\sigma = \sqrt{\sigma^2}$ is $\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2}$.

(b) Since $\text{Var}(S^2) = 2\,\sigma^4/(n\text{-}1)$, then by the invariance property of MLEs, the MLE is $2\,(S')^4/(n\text{-}1)$.

8. (Exercise 9.97, 1 pt)

a. MOM estimates the first population moment EY with the first sample moment $\overline{Y}$. Since $EY = 1/p$, then the MOM estimator for p is $\hat{p} = 1/\overline{Y}$.

b. We did this in class. The likelihood function is $L(p) = p^n(1-p)^{\Sigma y_i - n}$ and the log–likelihood is

$$\ln L(p) = n\ln p + (\sum_{i=1}^{n} y_i - n)\ln(1-p).$$

Differentiating, we have $\frac{d}{dp}\ln L(p) = \frac{n}{p} - \frac{1}{1-p}(\sum_{i=1}^{n} y_i - n)$. Equating this to 0 and solving for p, we obtain the MLE $\hat{p} = 1/\overline{Y}$, which is the same as the MOM estimator found in part **a.** The 2nd derivative is $\frac{d^2}{dp^2}\ln L(p) = -\frac{n}{p^2} - \frac{1}{(1-p)^2}(\sum_{i=1}^{n} y_i - n)$. Since $y_i \geq 1$ for every i, then $\sum_{i=1}^{n} y_i \geq n$. Thus, $\frac{d^2}{dp^2}\ln L(p) < 0$ which shows that $\hat{p} = 1/\overline{Y}$, is a maximizer.

9. (1 pt)

If $Y \sim$ Geometric(p), then (from the back of the book), $\text{Var}(Y) = t(p) = (1-p)/p^2$. Problem #8b (Exercise 9.97b) showed that the MLE for p is $\hat{p} = 1/\overline{Y}$. Thus, by the invariance property of MLEs, the MLE of $\text{Var(Y)} = t(p) = (1-p)/p^2$ is $t(1/\overline{Y}) = \overline{Y}^2(1 - 1/\overline{Y}) = \overline{Y}^2 - \overline{Y}$.

10. (Exercise 9.94, 1pt, required for grad students, EXTRA CREDIT otherwise)

Let $\beta = t(\theta)$ so that $\theta = t^{-1}(\beta)$. If the likelihood is maximized at $\hat{\theta}$, then $L(\hat{\theta}) \geq L(\theta)$ for all θ. Define $\hat{\beta} = t(\hat{\theta})$ and denote the likelihood as a function of β as $L_1(\beta) = L(t^{-1}(\beta))$. Then, for any β,

$$L_1(\beta) = L(t^{-1}(\beta)) = L(\theta) \leq L(\hat{\theta}) = L(t^{-1}(\hat{\beta})) = L_1(\hat{\beta}).$$

So, the MLE of β is $\hat{\beta}$ and so the MLE of $t(\theta)$ is $t(\hat{\theta})$.