Solutions Project 9 (20 points)

1. (1 pt) Follow the proof as on page 509 of your textbook.

2. (Exercise 10.78) (7 pts)

a. Assuming $Y \sim N(\mu = 800, \sigma^2 = 1600)$, then the probability of interest is $P(Y > 1000) = P(Z > \frac{1000-800}{40}) = P(Z > 5) = 3 \times 10^{-7}$. It is unlikely that the force is greater than 1000 lbs. We can still bound P(Y > 1000) even without assuming normality: via Tchebysheff's Theorem (Theorem 3.14), $P(Y > 1000) = P(Z > 5) \le 1/5^2 = 0.04$.

b. 1. H_0 : $\mu = 800$ vs. H_a : $\mu > 800$

2. The assumptions to check are: i. Was a SRS of helmets used? ii. The 5% rule is satisfied since the population of helmets is conceivably infinite. iii. The sample size of n = 40 is large enough for use of the *t*-test without assuming anything about the distribution of the data.

3. The test statistic is $t = \frac{825-800}{\sqrt{2350/40}} = \frac{825-800}{48.5/\sqrt{40}} = 3.262.$

4. The *p*-value is calculated from a *t* distribution with 39 degrees of freedom. Thus, P(T > 3.262) < .0016. Or maybe you prefer using the rejection region, RR={T > 1.68}.

5. Since the p-value is less than 0.05, we reject H_0 .

6. The evidence suggests that when subjected to a standard external force, the mean force transmitted is in excess of 800 lbs.

c. 1. H_0 : $\sigma^2 = 1600$ vs. H_a : $\sigma^2 > 1600$.

2. i. Was a SRS of helmets used? ii. The 5% rule is satisfied since the population of helmets is conceivably infinite. iii. The DATA MUST BE NORMAL regardless of sample size!

3. The test statistic is $\chi^2 = \frac{39(2350)}{1600} = 57.281$.

4. With 40 - 1 = 39 degrees of freedom, the p-value is $P(\chi^2 > 57.281) = 0.0296$. Or

maybe you prefer using the rejection region, RR = { $\chi^2 > \chi^2_{.05} = 54.57$ }.

5. Since the p-value is less than 0.05, then we can reject H_0

6. There is sufficient evidence to suggest that the true standard deviation of the forces that a helmet transmits to its wearer is in excess of 40.

3. (Exercise 10.80) (1 pt) Random samples must be taken from each population, the two samples must be independent of each other, and the two populations must be normal.

4. (Exercise 10.84) (3 pts) Let σ_1^2 , σ_2^2 denote the variances for measurements corresponding to 95% ethanol and 20% bleach, respectively.

1. The desired hypothesis test is $H_0: \sigma_1^2 = \sigma_2^2$ vs. $H_a: \sigma_1^2 \neq \sigma_2^2$

2. The assumptions to check are that i. SRS of experiments: were the experiments for each disinfectant performed independently of each other? ii. Were the sets of experiments for ethanol independent of the sets of experiments for bleach? iii. The 5% rule is satisfied since the population of experiments is conceivably infinite. iv. The results of the experiments for each

disinfectant MUST BE NORMALLY DISTRIBUTED regardless of sample size! Since the result for each experiment was a count between 0 and 5, then the normality assumption is NOT satisfied.

3. The computed test statistic is $F = \frac{2.78095}{0.17143} = 16.222.$

4. With 14 numerator and 14 denominator degrees of freedom, the *p*-value is $2P(F > 16.222) = 5.58970510^{-6}$. Or maybe you prefer using the rejection region, RR = {*F*|*F* < 0.336 or *F* > 2.98}.

5. Since the p-value is less than 0.05, we reject H_0

6. The evidence suggests that the variance of the number of uncontaminated eggplant cuttings after 4 weeks of storage is different for ethanol versus bleach.

5. (Exercise 10.91) (3 pts) In this problem, μ is unknown, σ^2 is known to be 5.

a. To find the <u>uniformly most powerful</u> test, first we must use the Neyman-Pearson Lemma to find the <u>most powerful</u> test for the set of simple hypotheses $H_0: \mu = 7$ vs. $H_a: \mu = \mu_a$ where $\mu_a > 7$. Example 10.23 in the text, which you are expected to mimic in your solutions for this problem set, shows that the most powerful test is the *z*-test from Section 10.3, with rejection region RR = { $Z = \frac{\overline{Y} - 7}{\sqrt{5/20}} > z_{.05} = 1.645$ } (or equivalently, reject if $\overline{Y} > 1.645\sqrt{.25} + 7 = 7.82$ }. Since the rejection region does not depend on the specific value of μ_a , then the *z*-test is in fact the uniformly most powerful test for the hypotheses $H_0: \mu = 7$ vs. $H_a: \mu = \gamma$.

b. The power function is: power(μ) = $P(\overline{Y} > 7.82 | \mu) = P(Z > \frac{7.82 - \mu}{\sqrt{5/20}})$. Thus:

power(7.5) =
$$P(\overline{Y} > 7.82 | 7.5) = P(Z > .64) = .2611.$$

power(8.0) = $P(\overline{Y} > 7.82 | 8.0) = P(Z > -.36) = .6406.$
power(8.5) = $P(\overline{Y} > 7.82 | 8.5) = P(Z > -1.36) = .9131$
power(9.0) = $P(\overline{Y} > 7.82 | 9.0) = P(Z > -2.36) = .9909.$



6. (2 pts, Exercise 10.100) Since **X** and **Y** are independent, the likelihood function is the product of all the marginal mass functions $L(y_1, ..., y_n | \lambda_1, \lambda_2) \propto \lambda_1^{\sum y_i} \lambda_2^{\sum x_i} \exp(-\lambda_1 n - \lambda_2 m)$. By the Neyman-Pearson Lemma, the most powerful test for the hypotheses H_0 : $\lambda_1 = \lambda_2 = 2$ vs. H_a : $\lambda_1 = \frac{1}{2}, \lambda_2 = 3$ is given by the rejection region RR=

$$\{\lambda \mid \lambda = \frac{L(\lambda_1 = \lambda_2 = 2)}{L(\lambda_1 = 1/2, \lambda_2 = 3)} = \frac{2^{\sum x_i + \sum y_i} \exp(-2m - 2n)}{\left(\frac{1}{2}\right)^{\sum y_i} 3^{\sum x_i} \exp(-n/2 - 3m)} = 4^{\sum y_i} \left(\frac{2}{3}\right)^{\sum x_i} \exp(-3n/2 + m) < k\}.$$

Taking logs

$$RR = \{ W = (\ln 4) \sum_{i=1}^{n} y_i + \ln(2/3) \sum_{i=1}^{m} x_i = (\ln 4) n\overline{y} + \ln(2/3) m\overline{x} < k' \}$$

a linear combination of independent Poisson rvs. If both *m* and *n* are large then \overline{x} and \overline{y} are both approximately N(2,2) (by CLT) in which case *W*, a linear combination of \overline{x} and \overline{y} , is also approximately N(EW, VW). Because

$$EW = (\ln 4)n\lambda_1 + \ln(2/3)m\lambda_2 = n\ln 16 + m\ln(4/9)$$

and

$$VW = [(n \ln 4)]^2 \lambda_1 + [m \ln(2/3)]^2 \lambda_2 = 2[[(n \ln 4)]^2 + [m \ln(2/3)]^2]$$

Thus, to maintain an α -level test, k' can be chosen as the α percentile

$$k' = n \ln 16 + m \ln(4/9) - z_{\alpha} \sqrt{2} \left[\left[(n \ln 4) \right]^2 + \left[m \ln(2/3) \right]^2 \right]^{1/2}.$$

This shows that the most powerful test has

$$\mathbf{RR} = \{W|W < n\ln 16 + m\ln(4/9) - z_{\alpha}\sqrt{2}\left[\left[(n\ln 4)\right]^2 + \left[m\ln(2/3)\right]^2\right]^{1/2}.$$

7. (Exercise 10.102) (3 pts)

a. The likelihood function is the product of the probability mass functions:

$$L(p) = p^{\Sigma y_i} (1-p)^{n-\Sigma y_i}.$$

i. It follows that the likelihood ratio is

$$\lambda = \frac{L(p_0)}{L(p_a)} = \frac{p_0^{\Sigma y_i} (1 - p_0)^{n - \Sigma y_i}}{p_a^{\Sigma y_i} (1 - p_a)^{n - \Sigma y_i}} = \left(\frac{p_0 (1 - p_a)}{p_a (1 - p_0)}\right)^{\Sigma y_i} \left(\frac{1 - p_0}{1 - p_a}\right)^n.$$

ii. Simplifying the above, the test rejects when

$$\ln(\lambda) = \sum_{i=1}^{n} y_i \ln\left(\frac{p_0(1-p_a)}{p_a(1-p_0)}\right) + n \ln\left(\frac{1-p_0}{1-p_a}\right) < \ln k.$$

Equivalently, this is

$$\sum_{i=1}^{n} y_{i} > \left[\ln k - n \ln \left(\frac{1 - p_{0}}{1 - p_{a}} \right) \right] \times \left[\ln \left(\frac{p_{0}(1 - p_{a})}{p_{a}(1 - p_{0})} \right) \right]^{-1} = k *$$

Note that the inequality sign changes from a "<" to a ">" because both sides of the inequality were divided by $\ln\left(\frac{p_0(1-p_a)}{p_a(1-p_0)}\right)$ which is negative because $p_0 < p$, which

implies that $p_0/p_a < 1$ and $(1 - p_a)/(1 - p_0) < 1$, and so $\frac{p_0(1 - p_a)}{p_a(1 - p_0)} < 1$ which shows

that
$$\ln\left(\frac{p_0(1-p_a)}{p_a(1-p_0)}\right) < 0$$

iii. The rejection region is of the form { $\sum_{i=1}^{n} y_i > k^*$ }.

b. Under H_0 , $\sum_{i=1}^{n} Y_i \sim \text{Bin}(n, p = p_0)$. Thus, to maintain the probability of a Type I error at α , choose the critical value k^* to be the 100(1- α)% percentile from Bin $(n, p = p_0)$, so that $P(\sum_{i=1}^{n} Y_i > k^* | p_0) = \alpha$, resulting in rejection region RR={ $\sum_{i=1}^{n} Y_i > k^*$ }. **c.** Since the critical value and rejection region can be specified without regard to a specific value of p_a , this is the UMP test.