KEY

Exam 2

MATH 221-2

March 31, 2014 Chapters 1-4

1. (28 pts, 2 pts each) Circle T or F indicating whether each of the following statements are True or False. You do not need to justify your answers.

(a) T F For any matrix A, $\mathcal{C}_A$ and $\mathcal{N}(A^T)$ are orthogonal vector spaces.

(b) T F If $\dim(\mathcal{N}(A)) = 1$, then $A\boldsymbol{x} = \boldsymbol{0}$ has an infinite number of solutions.

(c) T F If $A = \begin{bmatrix} | & | & & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_n \\ | & | & & | \end{bmatrix}$ then $\text{span}(\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n) = \mathcal{C}_A$.

(d) T F If A is a 7×8 matrix then A^{-1} is a 8×7 matrix.

(e) T F If two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, then $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent.

(f) T F For a 2×3 matrix A, if the general solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (z is a scalar), then there are non-zero solutions to $A^T\boldsymbol{y} = \boldsymbol{0}$.

(g) T F If $A = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 0 & 0 \\ 2 & 1 & -6 \end{bmatrix}$ then $\dim(\mathcal{C}_A) = 2$.

(h) T F If $A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix}$ then $A\boldsymbol{x} = \boldsymbol{b}$ has an infinite number of solutions for any $\boldsymbol{b} \in \mathcal{C}_A$.

(i) T F Any set of n linearly independent vectors in $\Re^n$ is a basis for $\Re^n$.

(j) T F If $\mathcal{V} = \text{span}(\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n)$ then $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ are a basis for $\mathcal{V}$.

(k) T F If the columns of A are linearly independent, then there is always at least 1 solution to $A\boldsymbol{x} = \boldsymbol{b}$.

(l) T F If A is a 3×2 matrix and $A\boldsymbol{x} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ has only one solution then $\mathcal{N}(A) = \{0\}$.

(m) T F If A is a 10×5 matrix, then $\text{rank}(A) = 10 - \dim(\mathcal{N}(A^T))$.

(n) T F If $\mathcal{N}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$, then $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in \mathcal{R}_A$.

2. (4 pts) Perform the following matrix-vector multiplication: $A\boldsymbol{x} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = ?$

$$A\underset{\sim}{x} = \begin{pmatrix} -1+2+3 \\ 0+2+0 \\ 2+2+6 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 10 \end{pmatrix}.$$

3. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \end{pmatrix}$.

(a) (16 pts) Solve the system of linear equations

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 4 \\ 2x & + & 4y & & & = & 2 \end{array}.$$

SHOW YOUR WORK.

the augmented matrix is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 0 & 2 \end{pmatrix}$$

$R_2 \leftarrow R_2 - 2R_1$

$$\begin{pmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & \boxed{-6} & -6 \end{pmatrix}$$

thus $-6z = +6 \Rightarrow \boxed{z = 1}$

Since there is no pivot for y, let $\boxed{y \text{ be free}}$.

Now solving for x via the 1st row shows that:

$x + 2y + 3z = 4 \Rightarrow x = -2y - 3z + 4$

$\boxed{x = 1 - 2y}$

(b) (5 pts) Based on the work you did in #3a, give a basis for C_A. Justify your answer.

Since pivots are in 1st and 3rd columns, then 1st & third columns of A form a basis for C_A.

that is

$$\boxed{C_A = \text{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}\right).}$$

(c) (5 pts) Based on your answer to #3a, give a basis for $\mathcal{N}(A)$. Justify your answer.

In (a), the general solution to $A\underline{x} = \underline{b}$ was

$$\underline{x} = \begin{pmatrix} 1-2y \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \underbrace{y\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}_{\text{homogeneous solutions in } N(A).}$$

This shows that $\boxed{N(A) = \text{span}\left(\left(\begin{smallmatrix} -2 \\ 1 \\ 0 \end{smallmatrix}\right)\right).}$

(d) (5 pts) Based on your answer to #3a, is $\begin{bmatrix} 4 \\ 2 \end{bmatrix} \in C_A$? Explain why your answer is correct.

Yes, since we found a solution to $A\underline{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, then $\begin{pmatrix} 4 \\ 2 \end{pmatrix} = (1-2y)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} 2 \\ 4 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is a l.c. of columns of A; that is $\begin{pmatrix} 4 \\ 2 \end{pmatrix} \in C_A$.

(e) (5 pts) For any non-zero $\boldsymbol{b} \in \Re^2$ that one might choose, explain why finding only 1 solution to $A\boldsymbol{x} = \boldsymbol{b}$ is impossible.

Because has a nullspace, $N(A) = \text{span}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, then any solution to $A\underline{x} = \underline{b}$ will always have a homogeneous component; that is, if $\underline{b} \in C_A$, there will be an infinite # of solutions to $A\underline{x} = \underline{b}$.

(f) (5 pts) For any $\boldsymbol{b} \in \Re^2$ that one might choose, explain why finding no solutions to $A\boldsymbol{x} = \boldsymbol{b}$ is impossible.

In #3b, we showed that $C_A = \mathbb{R}^2$. Thus, any $\underline{b} \in \mathbb{R}^2$ must be in C_A, so $A\underline{x} = \underline{b}$ always has solutions for any $\underline{b}$.

(g) (5 pts) Based on your answers to #3e-3f, for any $\boldsymbol{b} \in \Re^2$, how many solutions will there always be for $A\boldsymbol{x} = \boldsymbol{b}$? Explain.

$A\underline{x} = \underline{b}$ always has an infinite number of solutions because $\dim(N(A)) = 1$ and $C_A = \mathbb{R}^2$.

4. (12 pts) This is a 3-part question. Answer all 3 parts to get full credit: (i) What is a homogeneous solution of a matrix A? (ii) How do homogeneous solutions of A relate to the $\mathcal{N}(A)$? (iii) Why are homogeneous solutions important when solving $Ax = b$?

(i) If $\underline{x}$ is a homogenous solution, then $A\underline{x} = \underline{0}$

(ii) $N(A)$ contains all homogeneous solutions.

(iii) Homogeneous solutions are important because if they exist, then when solving $A\underline{x} = \underline{b}$, we will expect to get either 0 or ∞ number of solutions.

5. (10 pts) For any $m \times n$ matrix A, prove that $\mathcal{N}(A)$ is a vector space. Explicitly state the conditions that you are checking, and justify each step in your proof.

We must check (M1) (closure wrt scalar multiplication)
and (A1) (closure wrt vector addition)

To check (M1), let $\underline{u} \in N(A)$ and $\alpha \in \mathbb{R}$. Since

$$A(\alpha \underline{u}) = \underbrace{\alpha A\underline{u}}_{\text{because } \alpha \text{ is a scalar}} = \underbrace{\alpha \underline{0}}_{\text{because } A\underline{u} = \underline{0}} = \underline{0}$$

then $\alpha \underline{u} \in N(A)$. ✓

To check (A1), let $\underline{u}, \underline{v} \in N(A)$. Since

$$A(\underline{u} + \underline{v}) = \underbrace{A\underline{u} + A\underline{v}}_{\text{distributive property}} = \underbrace{\underline{0} + \underline{0}}_{\text{because } A\underline{u} = A\underline{v} = \underline{0}} = \underline{0}$$

then $\underline{u} + \underline{v} \in N(A)$.