Summary of Series Convergence/Divergence Theorems

A partial sum s_n is defined as the sum of terms of a sequence $\{a_k\}$. For instance,

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

If the limit of the partial sums converge as $n \to \infty$ then we say the infinite series converges and write:

$$s = \sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$$

If the limit $\lim_{n\to\infty} s_n$ does not exist then we say the infinite series $\sum_{k=1}^{\infty} a_k$ diverges.

(GST) GEOMETRIC SERIES TEST If $a_n = a\ r^{n-1}$ then the series $\sum_{n=1}^{\infty} a_n$ is a geometric series which converges only if $|r| < 1$ in which case

$$\sum_{n=1}^{\infty} a\ r^{n-1} = \sum_{n=0}^{\infty} a\ r^n = a + ar + ar^2 + \cdots = \frac{a}{1-r} \quad , \quad |r| < 1$$

If $|r| \geq 1$ the geometric series diverges.

(PST) P-SERIES TEST If $p > 1$ the series $\sum \frac{1}{n^p}$ converges. If $p \leq 1$ the series diverges.

(DT) DIVERGENCE TEST If $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.

(IT) INTEGRAL TEST Suppose $a_n = f(n)$ where $f(x)$ is a positive continuous function which decreases for all $x \geq 1$. Then

$$\sum_{n=1}^{\infty} a_n \quad and \quad \int_1^{\infty} f(x)dx$$

either both converge or both diverge. The same holds if $f(x)$ decreases for all $x > b$ where $b \geq 1$ is any number.

(CT) COMPARISON TEST Let N be some integer $N \geq 1$. Then

1) if $0 < a_n \leq b_n\ for\ all\ n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges

2) if $0 < b_n \leq a_n\ for\ all\ n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges

(LTC) LIMIT COMPARISON TEST

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$

Then either both series $\sum a_n$ and $\sum b_n$ converge or they both diverge.

(AST) Alternating Series Test Suppose

$$\begin{aligned} a_n &> 0 \\ a_{n+1} &\leq a_n \quad \textit{for all } n \\ \lim_{n\to\infty} a_n &= 0 \end{aligned}$$

then the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$$

converges.

(ASET) Alternating Series Estimation Theorem Suppose $s = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is a convergent alternating series satisfying the hypotheses of (AST) above. Then the difference between the n^{th} partial sum and s is at most the value of the next term a_{n+1}, i.e.,

$$\left|\sum_{n=1}^{\infty}(-1)^{n-1}a_n - s_n\right| = |s - (a_1 - a_2 + a_3 - a_4 + \cdots + a_n)| \leq a_{n+1}$$

The last two tests for series convergence require the following definitions:

If $\sum |a_n|$ converges then $\sum a_n$ is said to be absolutely convergent.

If $\sum a_n$ converges but $\sum |a_n|$ diverges then $\sum a_n$ is said to be conditionally convergent.

Note that (by a Theorem) every absolutely convergent series is convergent.

(RT) Ratio Test

1) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\sum a_n$ converges absolutely (and hence converges)

2) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ then $\sum a_n$ diverges
$= \infty$ then $\sum a_n$ diverges

3) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \mathrm{L} = 1$ then the test fails and nothing can be said

(RooT) Root Test

1) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$ then $\sum a_n$ converges absolutely (and hence converges)

2) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ then $\sum a_n$ diverges
$= \infty$ then $\sum a_n$ diverges

3) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \mathrm{L} = 1$ then the test fails and nothing can be said