Overview of Multivariate Integration

Final Sections: 15.2-15.4, 15.6-15.8, 16.1-16.3, 16.6-16.7 (graphs only), 16.9

DOUBLE INTEGRALS : Cartesian, Polar

Iterated integrals in cartesian coordinates using via x-slice and y-slice are, respectively:

$$\iint_R f(x,y)\ dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dydx = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dxdy \tag{1}$$

Iterated integrals in cartesian coordinates using via polar coordinates

$$\iint_R f(x,y) dA = \int_{\theta_1}^{\theta_2} \int_{R_1(r)}^{R_2(r)} f(r\cos\theta, r\sin\theta)\ rdrd\theta \tag{2}$$

Area elements

$$dA = dydx = dxdy = rdrd\theta$$

Coordinate conversions

$$\begin{array}{ll} x = r\cos\theta & y = r\sin\theta \quad , \\ r = \sqrt{x^2+y^2} & \theta = arctan(y/x) \end{array}$$

VOLUME INTEGRALS: Cartesian, Cylindrical, Spherical over solid E

Cartesian: One ordering is $dV = dzdA = dzdydx$

$$\iiint_E f(x,y,z)\ dV = \iiint_E f(x,y,z)\ dzdA = \int_a^b \int_{c(x)}^{d(x)} \int_{F_B(x,y)}^{F_T(x,y)} f(x,y,z)\ dzdydx \tag{3}$$

Cylindrical $\theta \in [0, 2\pi), r > 0$

$$\iiint_E f(x,y,z)\ dV = \int_{\theta_1}^{\theta_2} \int_{R_1(r)}^{R_2(r)} \int_{F_B(r,\theta)}^{F_T(r,\theta)} f(r\cos\theta, r\sin\theta, z)\ rdzdrd\theta \tag{4}$$

Spherical $\theta \in [0, 2\pi), \phi \in [0, \pi], \rho > 0$

$$\iiint_E f(x,y,z)\ dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1(\theta)}^{\phi_2(\theta)} \int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \rho^2\sin\phi\ d\rho d\phi d\theta \tag{5}$$

Coordinate System	dV	conversion formulae	notes
Cartesian	$dzdydx$	*	
Cylindrical	$r\ dzdrd\theta$	$x = r\cos\theta$ $y = r\sin\theta$	$r^2 = x^2 + y^2$
Spherical	$\rho^2 \sin\phi\ d\rho d\phi d\theta$	$x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$	$\rho^2\sin^2\phi = x^2 + y^2$

LINE INTEGRALS

Let $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} =< x(t), y(t), z(t) >$ with $t \in (t_1, t_2)$ be a parametrization of the curve C.

Line integrals of scalars $f(x, y, z)$

$$\int_C f(x,y,z)\ ds = \int_{t_1}^{t_2} f(x(t), y(t), z(t))\ |\vec{\mathbf{r}}\,'(t)|\ dt \tag{6}$$

Noting the arclength element ds is given by

$$ds = |\vec{\mathbf{r}}\,'(t)|\ dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt \tag{7}$$

If $f = 1$, the integral is the arclength of C. If f=mass density per unit length, the line integral is mass of C.

Line integrals of vector fields $\vec{\mathbf{F}}$

Let $\vec{\mathbf{F}} = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}} =< F_1, F_2, F_3 >$ be some vector field.

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} = \int_C F_1 dx + F_2 dy + F_3 dz = \int_{t_1}^{t_2} \vec{\mathbf{F}}(x(t), y(t), z(t)) \cdot \frac{d\vec{\mathbf{r}}}{dt}\ dt \tag{8}$$

If $\vec{\mathbf{F}}$ is force, the line integral is the work done by the force along path C.

SURFACE INTEGRALS: Scalars and Vector Fields

Let S be a surface and $\vec{\mathbf{N}}$ be normal to S. The normal vectors and surface elements dS are:

	Description	$\vec{\mathbf{N}}$	dS	Note:
Graph	$z = f(x,y)$	$< -f_x, -f_y, 1 >$	$\lvert\vec{\mathbf{N}}\rvert\ dA = \sqrt{1 + f_x^2 + f_y^2}\ dA$	On final
Parametrized surface	$\vec{\mathbf{r}}(u,v)$	$\frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}$	$\lvert\vec{\mathbf{N}}\rvert\ dudv = \lvert\frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}\rvert\ dudv$	Not on final

Scalar surface integrals of $G(x,y,z)$ **on a graph** $z = f(x,y)$

$$\iint_S G(x,y,z) dS = \iint_R F(x,y,f(x,y))\ |\vec{\mathbf{N}}|\ dA \tag{9}$$

If $F = 1$, the surface integral is the surface area of S. If F =mass density per unit area, the surface integral is the mass of S.

Vector surface integrals of $\vec{\mathbf{F}}$ thru the graph $z = f(x,y)$

Let S be an oriented surface, i.e., where the unit normal $\hat{\mathbf{N}}$ has been uniquely specified.

$$\Phi = \iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} = \iint_S \vec{\mathbf{F}} \cdot \hat{\mathbf{N}} dS = \iint_R \vec{\mathbf{F}}(x,y,f(x,y)) \cdot \vec{\mathbf{N}}\ dA \tag{10}$$

If $\vec{F} = \rho\vec{v}$ where ρ is fluid density, and $\vec{v}$ is the velocity field of the fluid, then the flux is the net rate of mass flow through S per unit time.

DIVERGENCE (GAUSS) THEOREM

Let V be some solid region in space and $S = \partial V$ be its bounding <u>closed</u> surface. If $\hat{\mathbf{N}}$ is the <u>outward</u> unit normal to the surface then

$$\iint_{\partial V} \vec{\mathbf{F}} \cdot \hat{\mathbf{N}}\ dS = \iiint_V \vec{\nabla} \cdot \vec{\mathbf{F}} dV \quad , \quad \vec{\nabla} \cdot \vec{\mathbf{F}} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{11}$$

STOKES THEOREM

Let S be some oriented surface with unit normal $\hat{\mathbf{N}}$ and boundary curve $C = \partial S$. Then for any (smooth) vector field $\vec{\mathbf{F}}$ we have

$$\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \hat{\mathbf{N}}\, dS = \int_{\partial S} \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} \tag{12}$$

says that certain surface integrals can be converted to line integrals and vice versa. Here, the curl of $\vec{\mathbf{F}}$ is:

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{13}$$

Green's Theorem Is the special case of Stokes Theorem where the surface S is a region in the xy-plane, i.e. $S = R$ and therefore $\hat{\mathbf{N}} = \hat{\mathbf{k}}$.