CHAPTERS 12-13 Review questions

(1) VECTOR CALCULATIONS

Let $\mathbf{u} =< 1, -2, 3 >, \mathbf{v} =< 3, 1, -4 >$ and $\hat{\mathbf{n}}$ be any unit vector. When the following expression makes sense compute it, otherwise state why the expression does not make sense. (here $\times$ is the cross product and $\cdot$ is the dot product)

a) $3\mathbf{u} - \mathbf{v}$ b) $\mathbf{u} \times \mathbf{v}$ c) $|\mathbf{u} \cdot \mathbf{v}|$

d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{u}$ e) $|\mathbf{v}|$ f) $|\mathbf{u} - \hat{\mathbf{n}}|^2 + 2\mathbf{u} \cdot \hat{\mathbf{n}}$

g) $\mathbf{u} \times |\mathbf{v}|$ h) $\hat{\mathbf{n}} \times \hat{\mathbf{n}}$ i) $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{v}$

(2) VECTORS-MISC

a) Find the angle between $\hat{\mathbf{i}} - \hat{\mathbf{k}}$ and $< 1, -1, 2 >$

b) What is the area of the triangle with vertices $P(1, 2, 0)$, $Q(1, 0, 2)$ and $R(0, 3, 1)$?

c) Find a unit vector orthogonal to $\mathbf{u} =< -1, 2, 0 >$ and $\mathbf{v} =< 1, 1, 1 >$

d) Find a vector parallel to $< 1, 4, 1 >$ and has the same length as $< 3, 0, 4 >$.

e) For $\mathbf{u} =< 1, -2, 3 >, \mathbf{v} =< 3, 1, -4 >$ find the projection of $\mathbf{u}$ onto $\mathbf{v}$, the projection of $\mathbf{v}$ onto $\mathbf{u}$ and the projection of $\mathbf{u}$ onto $\mathbf{u}$.

f) If $\mathbf{u} =< 0, 1, 3 >$ and $\mathbf{v} =< 1, 4, -2 >$ find $\mathbf{w}$ so that $2\mathbf{u} - \mathbf{v} + 3\mathbf{w} = \mathbf{0}$.

g) Find a vector parallel to $\mathbf{r}(t) =< 2 - t, 3t, 4 - 2t >$.

h) Find a vector perpendicular to $2x - y + z = 1$.

i) Find a vector perpendicular to $2x - y + z = 1$ and $x - y = 2$.

j) Find a vector which is perpendicular to the intersecting lines whose equations are $\mathbf{r}_1(t) =< 2-t, 3t, -1-2t >$ and $\mathbf{r}_2(t) =< 2 - 3t, t, 5t - 1 >$.

k) Given $\mathbf{v} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} = a\mathbf{e}_1 + b\mathbf{e}_2$, $\mathbf{e}_1 =< 2, 3 >$ and $\mathbf{e}_2 =< 1, -1 >$, find the scalars a and b.

l) Find a vector parallel to the curve $\mathbf{r}(t) =< t^2 - t, cos(t), t^3 >$ at time $t = 0$.

m) Compute the area of the triangle formed by the vectors $\vec{a} =< 1, 1, 1 >$ and $\vec{b} =< 1, 2, 3 >$.

(3) DISTANCE QUESTIONS

In each of the following, compute the distance between the indicated geometrical objects.

a) The points $P(1, 2, 4)$ and $Q(-1, -2, 1)$

b) The point $P(1, 2, 1)$ and the plane $3x - y + z = 0$.

c) The parallel lines $\mathbf{r}_1(t) =< 2 - t, 3t, 4 - 2t >$. and $\mathbf{r}_2(t) =< 3 - t, 3t + 1, 2 - 2t >$.

d) The parallel planes $2x - y + z = 2$ and $2x - y + z = 4$.

e) The point $P(1, 1, 1)$ and the line given by $\mathbf{r}(t) =< 3 - t, 3t + 1, 2 - 2t >$.

(4) INTERSECTION

a) Is the point $P(1,2,1)$ on the plane $2x - y + z = 1$?

b) Does the line given by $\mathbf{r}(t) =< 3 - t, 3t + 1, 2 - 2t >$ intersect the plane $x - y + z = 3$? If so, at what point?

c) Do the planes $x - 3y + 2z = 2$ and $2x - 6y + 4z = 7$ intersect? If so, what are the parametric equations for the line of intersection.

d) Do the planes $x - 3y + 2z = 2$ and $2x - 3y + 4z = 0$ intersect? If so, what are the parametric equations for the line of intersection.

e) At what point(s) does the curve $\mathbf{r}(t) =< 4 - t^2, 3t + 1, \sqrt{2+t} >$ intersect the $yz-$plane?

f) What is the intersection point of the lines given by:

$$\mathbf{r}_1(t) =< 2 - t, 3t, 4 - 2t > \quad , \quad \mathbf{r}_2(s) =< -1, 5, -1 > +s < 1, 1, 1 >$$

(5) LINES AND PLANES

a) Find an equation of a straight line through the points P(1,2,4) and Q(0,-1,2).

b) Find the equation of a straight line tangent to the curve given by $\mathbf{c}(t) =< t^2, 1 - t, cos(\pi t) >$ at $t = 1$.

c) Find the equation of a straight line perpendicular to the plane $x - 2y + z = 4$ and through the point P(1,0,1).

d) Find the equation of a plane with normal vector $\mathbf{N} =< 1, 4, -1 >$ through P(1,1,1).

e) Find the equation of a plane perpendicular to the curve $\mathbf{c}(t) =< t, t^2, t^3 >$ at $t = 1$.

f) Find the equation of a plane containing the points P(1,2,3), Q(0,1,1) and R(2,0,1).

g) Find the equation of the plane containing the two intersecting lines:

$$\mathbf{r}_1(t) =< 1, 2, 3 > +t < 1, -1, 1 > \quad , \quad \mathbf{r}_2(s) =< 1, 2, 3 > +s < 2, 0, 1 >$$

h) Find the equation of the plane which is normal to the planes $x + y - z = 1$ and $x - z = 2$ and contains the point $P(-1, -1, 2)$.

i) Two lines L_1 and L_2 having vector equations $\mathbf{r}_1(t) =< t, 2t - 2, -2 + 3t >$ and $\mathbf{r}_2(s) =< -1 + s, s - 2, -1 + s >$ intersect at a point P. Find the equation of a new line which passes through P and is perpendicular to the plane that contains L_1 and L_2.

(6) MISC. GEOMETRY

a) At what angle does the line given by $\mathbf{r}(t) =< 2t, 1 - t, 3 + 3t >$ intersect the plane $x + y + z = 0$?

b) What is (are) the angle(s) between the intersecting lines given by $\mathbf{r}_1(t) =< 0, 1, 2 > +t < 1, 2, 1 >$ and $\mathbf{r}_2(t) = t < 0, 1, 2 >$.

c) What is the center and radius of the sphere whose equation is $x^2 - 2x + y^2 - 4y + z^2 + 1 = 0$?

d) What is the distance from the sphere whose equation is $(x-2)^2 + (y-3)^2 + (z-1)^2 = 9$ and the point $P(1, 2, 1)$?

(7) ARCLENGTH

Compute the arclengths of the following curves on $0 \leq t \leq 1$:

a) $\mathbf{r}(t) =< 2+t, 1-t, 2+3t >$ b) $\mathbf{r}(t) =< cos(t), sin(t), 2+3t >$

c) $\mathbf{r}(t) =< cos(3t), sin(3t), 1-t >$ d) $\mathbf{r}(t) =< \cos(t)\sin(t), \sin^2(t), t >$

e) $\mathbf{r}(t) =< 3t\cos(t), 3t\sin(t), 2\sqrt{2}t^{3/2} >$ f) $\mathbf{r}(t) =< t+1, \frac{2}{3}(t+1)^{3/2}, \frac{1}{3}(2t+2)^{3/2} >$

g) $\mathbf{r}(t) =< \sqrt{t}, 3, \sqrt{t} >$ h) $\mathbf{r}(t) =< \frac{1}{\sqrt{2}}t^2, t, \frac{1}{3}t^3 >$

(8) CALCULUS OF CURVES

a) Find the velocity, speed and acceleration of a particle at time $t = 0$ if the position is:

i) $\mathbf{r}(t) =< 2+t, 1-t, 2+3t >$
ii) $\mathbf{r}(t) =< 3t\cos(t), 3t\sin(t), 2\sqrt{2}t^{3/2} >$
iii) $\mathbf{r}(t) =< cos(3t), sin(3t), 1-t >$

b) A curve is given by $\mathbf{r}(t) =< cos(3t), sin(3t), 1-t >$. Find unit tangent, [redacted].

c) [redacted].

d) For the following position vectors, decompose the acceleration into is normal and tangential components at $t = 0$ if $\mathbf{r}(t) =< 2cos(3t), 2sin(3t), t >$.

e) [redacted]:

[redacted]

CHAPTER 12-13 REVIEW SOLNS

QUESTION ONE

a) $\langle 0, -7, 13 \rangle$

b) $\langle 5, 13, 7 \rangle$

c) $|-1| = 1$

d) $\langle -11, 22, -33 \rangle$

e) $\sqrt{26}$

f) $(\vec{u}-\hat{n})\cdot(\vec{u}-\hat{n}) + 2\vec{u}\cdot\hat{n}$
$\vec{u}\cdot\vec{u} - \cancel{2\vec{u}\cdot\hat{n}} + \hat{n}\cdot\hat{n} + \cancel{2\vec{u}\cdot\hat{n}}$
$|\vec{u}|^2 + |\hat{n}|^2 = 14 + 1 = 15$

g) nonsense

h) $\hat{n} \times \hat{n} = \vec{0}$

i) nonsense.

QUESTION TWO

a) $\vec{u} = \langle 1, 0, -1 \rangle$ $\quad \vec{v} = \langle 1, -1, 2 \rangle$ $\quad \theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$

Calculations $\theta = \arccos\left(-\frac{1}{\sqrt{12}}\right)$

b) $P(1,2,0)$ $\; Q(1,0,2)$ $\; R(0,3,1)$ are vertices.

Area $= \frac{1}{2}|\vec{PQ} \times \vec{PR}| = \sqrt{6}$

where $\vec{PQ} = \langle 0, -2, 2 \rangle$ and $\vec{PR} = \langle -1, 1, 1 \rangle$.

c) $\vec{u} = \langle -1, 2, 0 \rangle$, $\vec{v} = \langle 1, 1, 1 \rangle$ hence $\vec{w} = \vec{u} \times \vec{v} = \langle 2, 1, -3 \rangle$

$|\vec{w}| = \sqrt{14}$ hence unit vect $\hat{w} = \frac{1}{\sqrt{14}}\langle 2, 1, -3 \rangle$

d) $\vec{u} = \langle 1, 4, 1 \rangle$ $\qquad |\vec{u}| = \sqrt{18}$

Hence unit vector in dir of $\vec{u}$ is $\hat{u} = \frac{1}{\sqrt{18}}\vec{u}$.

Desired vector $\vec{w}$ has length $|\langle 3, 0, 4 \rangle| = 5$.

$\vec{w} = 5\hat{u} = \frac{5}{\sqrt{18}}\vec{u}$.

e) $\text{proj}_v \vec{u} = \dfrac{(\vec{u}\cdot\vec{v})}{(\vec{v}\cdot\vec{v})}\vec{v} = -\dfrac{11}{26}\vec{v}$ $\quad \vec{u}$ onto $\vec{v}$

$\text{proj}_u \vec{v} = \dfrac{(\vec{u}\cdot\vec{v})}{(\vec{u}\cdot\vec{u})}\vec{u} = -\dfrac{11}{14}\vec{u}$ $\quad \vec{v}$ onto $\vec{u}$

$\text{proj}_u \vec{u} = \vec{u}$ for all $\vec{u}$.

f) $\vec{w} = \frac{1}{3}(\vec{v} - 2\vec{u}) = \langle \frac{1}{3}, \frac{2}{3}, -\frac{8}{3} \rangle$

g) $\vec{r} = \langle 2, 0, 4\rangle + \langle -1, 3, -2\rangle t$ $\qquad \vec{v} = \langle -1, 3, -2\rangle$

h) $\vec{N} = \langle 2, -1, 1\rangle$

i) $\vec{N}_1 = \langle 2, -1, 1\rangle \perp 2x - y + z = 1$

$\vec{N}_2 = \langle 1, -1, 0\rangle \perp x - y = 2$

$\vec{N} = \vec{N}_1 \times \vec{N}_2 = \langle 1, 1, -1\rangle$ $\perp$ both planes

j) $\vec{v}_1 = \langle -1, 3, 2\rangle \parallel$ line $\vec{r}_1(t)$

$\vec{v}_2 = \langle -3, 1, 5\rangle \parallel$ line $\vec{r}_2(t)$

$\vec{N} = \vec{v}_1 \times \vec{v}_2 = \langle 13, -1, 8\rangle$ $\perp$ intersecting lines

k) $\vec{v} = \langle 2, -1\rangle = \langle 2a+b, 3a-b\rangle = a\vec{e}_1 + b\vec{e}_2$

$$\left.\begin{array}{l} 2a + b = 2 \\ 3a - b = -1 \end{array}\right\} \text{ solve } \quad a = \frac{1}{5} \quad b = \frac{8}{5}$$

l) $\vec{r}\,'(t) = \langle 2t - 1, -\sin t, 3t^2\rangle$ $\qquad \vec{r}\,'(0) = \langle -1, 0, 0\rangle$

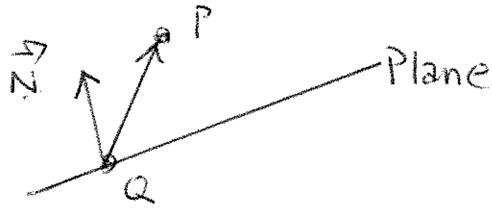
m) Area $A = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}|\langle 1, -2, 1\rangle| = \frac{1}{2}\sqrt{6}$

QUESTION 3 (Distance)

a) $\vec{PQ} = Q - P = \langle -2, -4, -3 \rangle \qquad |\vec{PQ}| = \sqrt{29}$

b) $\vec{N} = \langle 3, -1, 1 \rangle$ is $\perp$ plane. $3x - y + z = 0$

$Q(0, 1, 1)$ is a point on the plane.

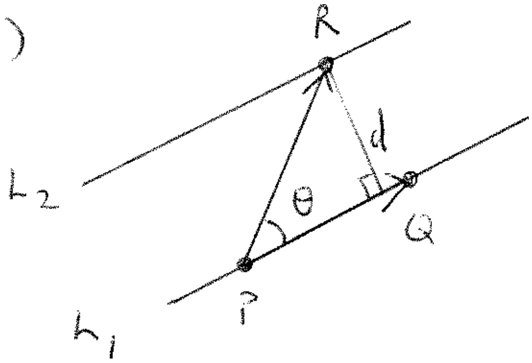


$\vec{QP} = P - Q$

$\vec{QP} = \langle 1, 1, 0 \rangle$

Distance $d = \left| \mathrm{comp}_{\vec{N}} \vec{QP} \right| = \dfrac{|\vec{N} \cdot \vec{QP}|}{|\vec{N}|} = \dfrac{2}{\sqrt{11}}$

c)



Pick any 3 points as indicated.

Say, for instance,

$P = \vec{r}_1(0) = (2, 0, 4)$
$Q = \vec{r}_1(1) = (1, 3, 2)$
$R = \vec{r}_2(0) = (3, 1, 2)$

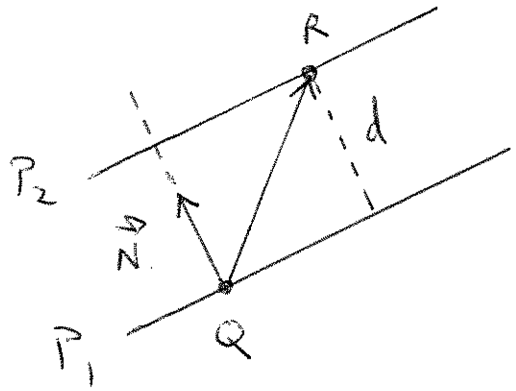
Thus $\vec{PQ} = \langle -1, 3, -2 \rangle$ and $\vec{PR} = \langle 1, 1, -2 \rangle$.

$$d = |\vec{PR}| \sin\theta = \frac{|\vec{PQ} \times \vec{PR}|}{|\vec{PQ}|}$$

So $\vec{PQ} \times \vec{PR} = \langle -4, -4, -4 \rangle$ and

$$d = \frac{|\langle -4, -4, -4 \rangle|}{|\langle -1, 3, -2 \rangle|} = \frac{\sqrt{48}}{\sqrt{14}} = 2\sqrt{\frac{6}{7}}.$$

d) Parallel Planes



$\vec{N} = \langle 2, -1, 1 \rangle \perp$ both planes

$Q(1, 0, 0)$ on $2x - y + z = 2$

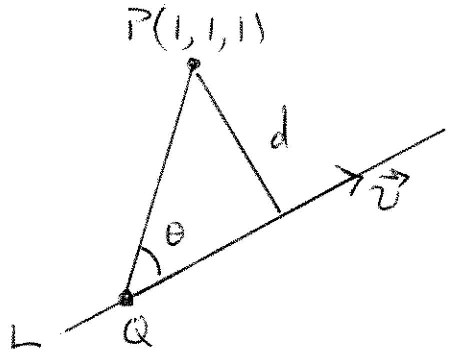
$R(0, 1, 5)$ on $2x - y + z = 4$

$\vec{QR} = \langle -1, 1, 5 \rangle$

Now question is very similar to 3b)

$$d = \left| \text{comp}_{\vec{N}} \vec{QR} \right| = \frac{|\vec{N} \cdot \vec{QR}|}{|\vec{N}|} = \frac{2}{\sqrt{6}}$$

e) Is similar to 3c)



$\vec{r}(t) = \langle 3, 1, 2 \rangle + t \langle -1, 3, -2 \rangle$

Let

$Q(3, 1, 2)$ on line.

$\vec{v} = \langle -1, 3, -2 \rangle$

$\vec{QP} = \langle 2, 0, 1 \rangle$

Indicated distance $d = |\vec{PQ}| \sin\theta$ so

$$d = \frac{|\vec{QP} \times \vec{v}|}{|\vec{v}|} = \frac{|\langle -3, 3, 6 \rangle|}{|\langle -1, 3, -2 \rangle|} = \frac{\sqrt{54}}{\sqrt{14}}$$

QUESTION 4 (Intersection)

a) yes since $2(1)-(2)+(1) = 1$

b) $x(t) - y(t) + z(t) = 4 - 6t = 3$ when $t = \frac{1}{6}$.

Intersection when $\vec{r}\left(\frac{1}{6}\right) = \frac{1}{6}\langle 17, 9, 10\rangle$.

c)
$$\vec{N}_1 = \langle 1, -3, 2\rangle \perp x - 3y + 2z = 2$$
$$\vec{N}_2 = \langle 2, -6, 4\rangle \perp 2x - 6y + 4z = 7$$

Since $\vec{N}_2 = 2\vec{N}_1$ planes are $\parallel$ and do not intersect.

d) Normal vectors $\vec{N}_1 = \langle 1, -3, 2\rangle$, $\vec{N}_2 = \langle 2, -3, 4\rangle$ are not parallel so planes intersect.

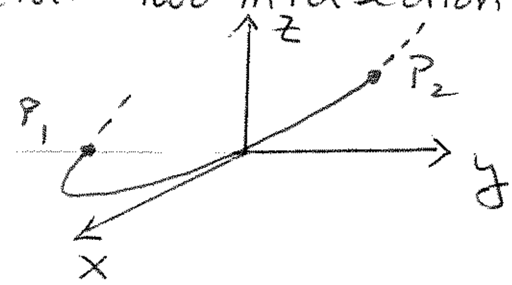
There are many different parametric equations. Here's one. Set $z(t) = t$ and solve for $x(t), y(t)$ in:

$$\begin{aligned} x - 3y + 2z &= 2 \\ 2x - 3y + 4z &= 0 \end{aligned} \qquad \left.\begin{aligned} x(t) &= -2 - 2t \\ y(t) &= -4/3 \\ z(t) &= t \end{aligned}\right\} \text{ on } y = -4/3 \text{ plane}$$

e) Curve intersects $x = 0$ or yz-plane when $x(t) = 4 - t^2 = 0$ or at $t = \pm 2$.

$$\vec{r}(2) = \langle 0, 7, 2\rangle \qquad \vec{r}(-2) = \langle 0, -5, 0\rangle$$

yields two intersection points



$P_1(0, -5, 0)$

$P_2(0, 7, 2)$

f) Intersecting Lines

$$\vec{r}_1 = \langle 2-t,\ 3t,\ 4-2t \rangle$$
$$\vec{r}_2 = \langle -1+s,\ 5+s,\ -1+s \rangle$$

Intersection point occurs for (s,t) pairs where <u>all</u> three coordinates equal

$$\left.\begin{aligned} x:&\quad 2-t = -1+s \\ y:&\quad 3t = 5+s \end{aligned}\right\}$$
$$z:\quad 4-2t = -1+s$$

Solve for $(s,t) = (1,2)$ and then verify also solves third eqn

The actual intersection point is found from either $\vec{r}_1(2) = \langle 0, 6, 0 \rangle$ or $\vec{r}_2(1) = \langle 0, 6, 0 \rangle$, i.e

$$P(0, 6, 0)$$

<u>is</u> the intersection point.

QUESTION FIVE

a) $\vec{r}_0 = \langle 1, 2, 4 \rangle$ $\quad \vec{v} = \overrightarrow{PQ} = \langle -1, -3, -2 \rangle$

$$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle 1-t, 2-3t, 4-2t \rangle$$

b) $\vec{c}\,'(t) = (2t, -1, -\pi \sin \pi t)$ $\quad \vec{c}\,'$

$\vec{r}_0 = \vec{c}(1) = \langle 1, 0, -1 \rangle$ $\quad \vec{v} = \vec{c}\,'(1) = \langle 2, -1, 0 \rangle$

$$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle 1+2t, -t, -1 \rangle$$

c) $\vec{N} = \langle 1, -2, 1 \rangle$ is $\perp$ $x - 2y + z = 4$.

$\vec{r}_0 = \langle 1, 0, 1 \rangle$ $\quad \vec{v} = \vec{N} = \langle 1, -2, 1 \rangle$

$$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle 1+t, -2t, 1+t \rangle$$

d) $\vec{N} = \langle 1, 4, -1 \rangle$ $\quad \vec{r}_0 = \langle 1, 1, 1 \rangle$

$$\vec{N} \cdot (\vec{r} - \vec{r}_0) = 0 \iff x - 4y - z = -4$$

e) $\vec{c}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$\vec{N} = \vec{c}\,'(1) = \langle 1, 2, 3 \rangle$ $\quad \vec{r}_0 = \vec{c}(1) = \langle 1, 1, 1 \rangle$

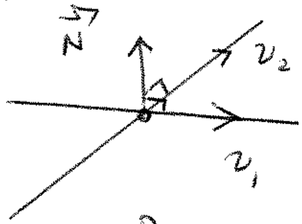
$$\vec{N} \cdot (\vec{r} - \vec{r}_0) = 0 \iff x + 2y + 3z = 6$$

f) $\overrightarrow{PQ} = \langle -1, -1, -2 \rangle$ $\quad \overrightarrow{PR} = \langle 1, -2, -2 \rangle$

$\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle -2, -4, 3 \rangle$ $\quad \vec{r}_0 = \overrightarrow{OP} = \langle 1, 2, 3 \rangle$

$$\vec{N} \cdot (\vec{r} - \vec{r}_0) = 0 \iff -2x - 4y + 3z = -1$$

g) $\vec{v}_1 = \langle 1, -1, 1 \rangle$ and $\vec{v}_2 = \langle 2, 0, 1 \rangle$; $\vec{r}_0 = \langle 1, 2, 3 \rangle$ on plane



$$\vec{N} = \vec{v}_1 \times \vec{v}_2 = \langle -1, 1, 2 \rangle$$

$$\vec{N} \circ (\vec{r} - \vec{r}_0) = 0 \iff -x + y + 2z = 7$$

h) $\vec{N}_1 = \langle 1, 1, -1 \rangle$ is $\perp$ plane $x + y - z = 1$
$\vec{N}_2 = \langle 1, 0, -1 \rangle$ is $\perp$ plane $x - z = 2$.

Plane we seek has $\vec{N} = \vec{N}_1 \times \vec{N}_2 = \langle -1, 0, -1 \rangle$
and $\vec{r}_0 = \vec{OP} = \langle -1, -1, 2 \rangle$

$$\vec{N} \circ (\vec{r} - \vec{r}_0) = 0 \iff x + z = 1$$

i) Intersection Point P from $\vec{r}_1(t) = \vec{r}_2(s)$

$$\left.\begin{array}{l} t = -1 + s \\ 2t - 2 = s - 2 \end{array}\right\} \text{ yields } (s,t) = (1,2)$$
$$-2 + 3t = -1 + s \qquad \vec{OP} = \langle 1, 0, 1 \rangle \text{ Pt on line}$$

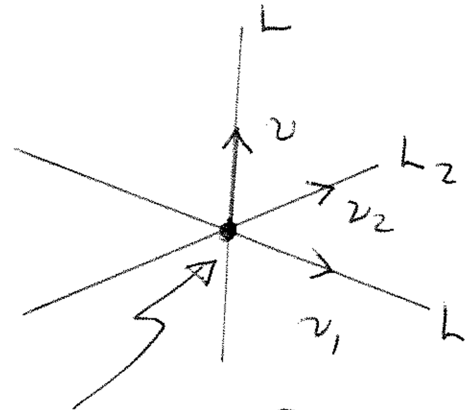
Then direction of line

$\vec{v}_1 = \langle 1, 2, 3 \rangle$ $\perp$ line L_1
$\vec{v}_2 = \langle 1, 1, 1 \rangle$ $\perp$ line L_2

$$\vec{v} = \vec{v}_1 \times \vec{v}_2 = \langle -1, 2, -1 \rangle$$

$$\vec{r}(t) = \vec{r}_0 + t\,\vec{v}$$

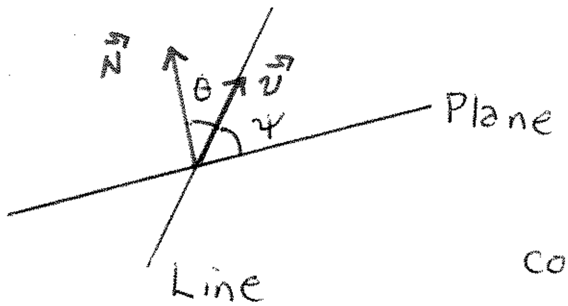
$$\vec{r}(t) = \langle 1 - t, 2t, 1 - t \rangle$$



$\vec{OP} = \langle 1, 0, 1 \rangle$ Int. Pt.

QUESTION 6

a)



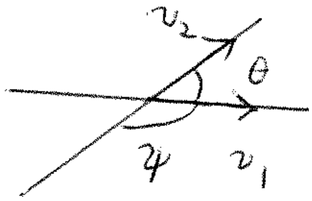
$\vec{N} = \langle 1, 1, 1 \rangle \qquad |\vec{N}| = \sqrt{3}$

$\vec{v} = \langle 2, -1, 3 \rangle \qquad |\vec{v}| = \sqrt{14}$

$$\cos\theta = \frac{\vec{N} \cdot \vec{v}}{|\vec{N}|\,|\vec{v}|} = \frac{4}{\sqrt{42}}$$

$$\text{Angle } \psi = \frac{\pi}{2} - \arccos\left(\frac{4}{\sqrt{42}}\right)$$

b)



$$\cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1|\,|\vec{v}_2|} = \frac{4}{\sqrt{30}}$$

$\vec{v}_1 = \langle 1, 2, 1 \rangle$

$\vec{v}_2 = \langle 0, 1, 2 \rangle$

$\theta = \arccos\left(\frac{4}{\sqrt{30}}\right)$ one angle

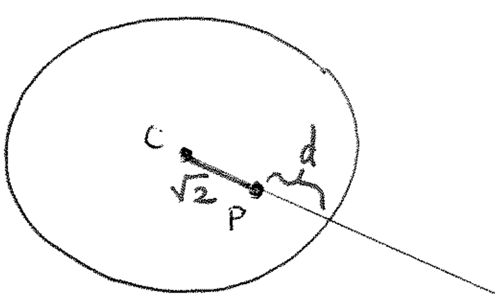
$\psi = \pi - \theta$ other angle.

c) Complete the square.

$$(x-1)^2 + (y-2)^2 + z^2 = 4$$

Hence radius $r = 2$, center $C(1, 2, 0)$.

d) Sphere has center $C(2,3,1)$ and radius $r = 3$. A quick calculation shows the distance $|PC| = \sqrt{2} <$ radius $= 3$ hence picture below showing point C inside sphere



$\vec{PC} = \langle 1, 1, 0 \rangle$

$d = r - |PC|$

$d = 3 - \sqrt{2}$

QUESTION 7

a) $\vec{v} = \langle 1, -1, 3 \rangle \qquad |\vec{v}|^2 = 11$

$$L = \int_0^1 |\vec{v}|\,dt = \int_0^1 \sqrt{11}\,dt = \sqrt{11}$$

b) $\vec{v} = \langle -\sin t, \cos t, 3 \rangle \qquad |\vec{v}|^2 = 10$ (trig ident)

$$L = \int_0^1 |\vec{v}|\,dt = \int_0^1 \sqrt{10}\,dt = \sqrt{10}$$

c) $\vec{v} = \langle -3\sin(3t), 3\cos 3t, -1 \rangle \qquad |\vec{v}|^2 = 10$ (trig)

$$L = \int_0^1 |\vec{v}|\,dt = \int_0^1 \sqrt{10}\,dt = \sqrt{10}$$

d) $\vec{v} = \langle \cos^2 t - \sin^2 t, 2\sin t \cos t, 1 \rangle = \langle \cos 2t, \sin 2t, 1 \rangle$

Thus $|\vec{v}|^2 = 2$ by trig ident

$$L = \int_0^1 |\vec{v}|\,dt = \int_0^1 \sqrt{2}\,dt = \sqrt{2}$$

e) $\vec{v} = \langle 3\cos t - 3t\sin t, 3\sin t + 3t\cos t, 3\sqrt{2}\sqrt{t} \rangle$

$|\vec{v}|^2 = \ldots$ calculations $= 18t + 9t^2 + 9 = 9(t+1)^2$

$$L = \int_0^1 \sqrt{9(t+1)^2}\,dt = \int_0^1 3(t+1)\,dt = 9/2$$

f) $\vec{v} = \langle 1, \sqrt{t+1}, \sqrt{2t+2} \rangle$

$|\vec{v}|^2 = 4 + 3t$

$$L = \int_0^1 |\vec{v}|\,dt = \int_0^1 \sqrt{4+3t}\,dt = \frac{1}{9}(14\sqrt{7} - 16)$$

g) $\vec{v} = \langle \frac{1}{2\sqrt{t}}, 0, \frac{1}{2\sqrt{t}} \rangle \qquad |\vec{v}|^2 = \frac{1}{2t}$

$$L = \int_0^1 |\vec{v}|\, dt = \int_0^1 \frac{1}{\sqrt{2t}}\, dt = \sqrt{2}$$

h) $\vec{v} = \langle \sqrt{2}\, t, 1, t^2 \rangle$

$$|\vec{v}|^2 = t^4 + 2t^2 + 1 = (t^2+1)^2$$

$$L = \int_0^1 |\vec{v}|\, dt = \int_0^1 (t^2+1)\, dt = 4/3$$

QUESTION 8

a) i) $\vec{r} = \langle 2+t, 1-t, 2+3t \rangle \qquad \vec{r}(0) = \langle 2, 1, 2 \rangle$

$\vec{v} = \langle 1, -1, 3 \rangle \qquad \vec{v}(0) = \langle 1, -1, 3 \rangle$

$\vec{a} = \langle 0, 0, 0 \rangle \qquad \vec{a}(0) = \langle 0, 0, 0 \rangle$

ii) $\vec{r} = \langle 3t\cos t, 3t\sin t, (2t)^{3/2} \rangle \qquad \vec{r}(0) = \langle 0, 0, 0 \rangle$

$\vec{v} = \langle 3\cos t - 3t\sin t, 3\sin t + 3t\cos t, 3\sqrt{2t} \rangle \qquad \vec{v}(0) = \langle 3, 0, 0 \rangle$

$\vec{a} = \langle -6\sin t - 3t\cos t, 6\cos t - 3t\sin t, \frac{3}{\sqrt{2t}} \rangle \qquad \vec{a}(0)$ undefined

iii) $\vec{r} = \langle \cos(3t), \sin(3t), 1-t \rangle \qquad \vec{r}(0) = \langle 1, 0, 1 \rangle$

$\vec{v} = \langle -3\sin(3t), 3\cos(3t), -1 \rangle \qquad \vec{v}(0) = \langle 0, 3, -1 \rangle$

$\vec{a} = \langle -9\cos(3t), -9\sin(3t), 0 \rangle \qquad \vec{a}(0) = \langle -9, 0, 0 \rangle$

b) Unit tangent, normal and bi-normal vectors

$$\vec{r} = \langle \cos(3t), \sin(3t), 1-t \rangle$$

$$\vec{r}\,' = \langle -3\sin(3t), 3\cos(3t), -1 \rangle \qquad |\vec{r}\,'| = \sqrt{10}$$

$$\vec{T}(t) = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \frac{1}{\sqrt{10}} \langle -3\sin(3t), 3\cos(3t), -1 \rangle$$

is the unit tangent.

$$\vec{T}' = \frac{9}{\sqrt{10}} \langle -\cos(3t), -\sin(3t), 0 \rangle \qquad |\vec{T}'| = \frac{9}{\sqrt{10}}$$

Thus a unit normal $\vec{N}$ is

$$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|} = \langle -\cos(3t), -\sin(3t), 0 \rangle$$

and the binormal vector is

$$\vec{B} = \vec{T} \times \vec{N} = \frac{1}{\sqrt{10}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin(3t) & 3\cos(3t) & -1 \\ -\cos(3t) & -\sin(3t) & 0 \end{vmatrix}$$

$$\vec{B} = \frac{1}{\sqrt{10}} \langle -\sin(3t), \cos(3t), 3 \rangle$$

c) Compute curvature using

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} \qquad \kappa(0) = \frac{|\vec{v}(0) \times \vec{a}(0)|}{|\vec{v}(0)|^3}$$

Here

$$\vec{r}'(t) = \vec{v}(t) = \langle e^t \sin 3t + 3e^t \cos 3t, 2t, -1 \rangle \qquad \vec{v}(0) = \langle 3, 0, -1 \rangle$$

$$\vec{r}''(t) = \vec{a}(t) = \langle -8t \sin 3t + 6e^t \cos 3t, 2, 0 \rangle \qquad \vec{a}(0) = \langle 6, 2, 0 \rangle$$

From which $|\vec{v}(0)| = \sqrt{10}$, $\vec{v}(0) \times \vec{a}(0) = \langle 2, -6, 6 \rangle$

$$\kappa(0) = \frac{|\langle 2, -6, 6 \rangle|}{10^{3/2}} = \frac{\sqrt{76}}{10^{3/2}}$$

d)

$$\vec{r}(t) = \langle 2\cos(3t), 2\sin(3t), t \rangle$$

$$\vec{v}(t) = \langle -6\sin(3t), 6\cos(3t), 1 \rangle \qquad \vec{v}(0) = \langle 0, 6, 1 \rangle$$

$$\vec{a}(t) = \langle -18\cos(3t), -18\sin(3t), 0 \rangle \qquad \vec{a}(0) = \langle -18, 0, 0 \rangle$$

acceleration $\vec{a} = a_T \hat{T} + a_N \hat{N}$ where

$$a_T(0) = \frac{\vec{v}(0) \cdot \vec{a}(0)}{|\vec{v}(0)|} = 0$$

$$a_N(0) = \frac{|\vec{v}(0) \times \vec{a}(0)|}{|\vec{v}(0)|} = \frac{|\langle 0, -18, 108 \rangle|}{|\langle 0, 6, 1 \rangle|} = 18$$

Note, in this case since $a_T(0) = 0$, $a_N = |\vec{a}(0)| = 18$.

e) i)

$$\int_0^1 \vec{r}(t)\,dt = \left\langle \int_0^1 (1+2t^2)\,dt, \int_0^1 (1-t)\,dt, \int_0^1 (2-3t)\,dt \right\rangle$$

$$= \left\langle \tfrac{5}{3}, \tfrac{1}{2}, \tfrac{1}{2} \right\rangle \quad \text{Vector!}$$

ii) For given $\vec{r}(t)$, $|\vec{r}|^2 = \langle \vec{r}, \vec{r} \rangle$

$$|\vec{r}|^2 = (1+2t^2) + (1-t)^2 + (2-3t)^2$$

$$= 4t^4 + 14t^2 - 14t + 6$$

So

$$\int_0^1 |\vec{r}|^2\,dt = \int_0^1 (4t^4 + 14t^2 - 14t + 6)\,dt$$

$$= \frac{67}{15} \quad \text{scalar.}$$