

Chapter 14: Review Questions

(1) PARTIAL DERIVATIVES AND IMPLICIT DIFFERENTIATION

- a) If $f(x, y) = y/(x + y)$ compute $f_x(2, -1)$ and $f_{xy}(2, -1)$.
- b) If $f(x, y, z) = x^3y + z^2y$ compute $f_{xz}(1, 3, 1)$ and $f_{yz}(1, 3, 1)$.
- c) If $f(x, y) = \sqrt{2x + 3y}$ compute $f_x(1, 1)$ and $f_y(1, 1)$.
- d) Compute $f_x(1, 0)$ if $f(x, y) = e^{x-1} \sin(xy)$.
- e) Compute $z_x(1, 1)$ if $z(x, y)$ is defined implicitly by $x^2z^3 - zy + x - 1 = 0$. Assume $z(x, y) \neq 0$.
- f) If $x^2 + z^4 - z - y^2 = 0$, compute z_x and z_{xx} in terms of x, y and z .

(2) LIMITS AND DOMAINS

- a) What are the domains of the following functions:
- i) $f(x, y) = \sqrt{1 - x - y}$ ii) $f(x, y) = \log(3 - x^2 - y^2)$ iii) $f(x, y) = (x - \sqrt{1 - y^2})^{-1}$
- b) Compute the following limits: *Pick any path such as $y = x$ etc.*
- i) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + xy - yx^2 - y^2}{x + x^2 - y - xy}$ ii) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - x + xy - y}{xy - 3x + y^2 - 3y}$ iii) $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{xy}$
- c) Show the following limits do not exist:

i) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{2x^2 + 3y^2}$ ii) $\lim_{(x,y) \rightarrow (1,2)} \frac{x - 3 + y}{x + 1 - y}$ iii) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4 + y^2}$

(3) CHAIN RULE

- a) Let $f = x^2 + y^2 + 3z$ where $x = 2t + z$ and $y = t - 3z$. Use the chain rule to evaluate f_z and f_t when $(z, t) = (1, 1)$.
- b) Let $w = x^2 - xy$, $x = u + v$, $y = u - v$. Use the chain rule to evaluate w_u when $(x, y) = (1, 0)$.
- c) You do not know $f(x, y)$ but do know $f_x = x^2 - y$ and $f_y = x + y$. If $x(t) = t^2 + t$ and $y(t) = 1 - 3t$, what is the rate of change of f in t when $t = 1$?

(4) GRADIENT AND DIRECTIONAL DERIVATIVES

- a) Compute $\nabla f(1, 2)$ if $f(x, y) = x^2 - \sqrt{y - x}$.
- b) What is the directional derivative of $f(x, y) = x^3y - y$ at $P(1, 1)$ in the direction $\mathbf{u} = \langle 1, 3 \rangle$. In what direction is f decreasing most rapidly at $(1, 1)$?
- c) What is the directional derivative of $f(x, y) = xy - y^2$ at $P(1, 1)$ in a direction toward $Q(4, 5)$?
- d) The temperature in a room is given by $f(x, y) = 100x/(x^2 + y^2)$ °F where x, y (feet) are coordinates on the floor. You move along $x = 1$ at 2 ft/sec in a positive y direction carrying a thermometer. At the moment you pass thru $(x, y) = (1, 1)$ at what rate is the temperature on your thermometer changing?
- e) ~~Let $f(x, y) = F(r)$ where $r = \sqrt{x^2 + y^2}$. Show~~

$$\nabla f = F'(r) \hat{\mathbf{r}}, \quad \mathbf{r} = \langle x, y \rangle = x \hat{\mathbf{i}} + y \hat{\mathbf{j}}, \quad \hat{\mathbf{r}} = \frac{1}{r} \mathbf{r} \quad (1)$$

(5) GEOMETRY

- Find two unit vectors perpendicular to the graph of $f(x, y) = x^2 + y^3$ at $(x, y) = (1, 1)$.
- Find two nonparallel vectors tangent to the graph of $f(x, y) = xy + y^2$ at $(x, y) = (1, 1)$.
- Find the equation of a plane tangent to the graph of $f(x, y) = x + y + xy^3$ at $(x, y) = (1, 1)$.
- Find the equation of a line, normal to the graph of $f(x, y) = \sqrt{x + 3y}$ at $(x, y) = (1, 1)$.
- A line L is normal to the graph of $f(x, y) = 1/(x + 2y + 1)$ at $(x, y) = (1, 1)$. At what point does this line intersect the xy -plane?
- Find a vector perpendicular to the curve $x^2y = 1$ when $(x, y) = (1, 1)$.
- Find a vector perpendicular to the surface defined by $x^2 - z^3 + zy = 0$ when $(x, y) = (1, 0)$.
- Find a unit vector perpendicular to the $f = 2$ level curve of $f(x, y) = x^2 + y^3$ at $(x, y) = (1, 1)$.
- A curve C is formed by the intersection of the cylinder $x^2 + y^2 = 1$ and the graph $z = f(x, y)$. If $\nabla f(x, y) = \langle x + y, x - y \rangle$, find a vector tangent to the curve when $(x, y) = (1, 0)$ (Hint: $x(t) = \cos t$).

(6) CRITICAL POINTS AND SECOND DERIVATIVE TEST

- Find all the critical points of f and classify them
 - $f(x, y) = x^2 + 2y^2 - 4x + 4y$ (2,-1) local minima
 - $f(x, y) = xy - x + y$ (-1,1) saddle
 - $f(x, y) = (2 - x)e^{x-y^2}$ (1,0) local maxima
 - $f(x, y) = x^4 + y^4 - 4xy$ (0,0) saddle
(1,1), (-1,-1) local minima
 - $f(x, y) = x/y + 8/x - y$ (-4,2) local maxima
 - $f(x, y) = x^3 + y^3 - 3xy$ (0,0) saddle, (1,1) local minima
 - $f(x, y) = x^2 - 3xy + 5x - 2y + 6y^2$ (-18/5, -11/15) local minima
- If $\nabla f = \langle f_x, f_y \rangle = \langle x^2 - y, y - x \rangle$, find and classify all critical points of $f(x, y)$.

(7) MAXIMA AND MINIMA ON A REGION

- Find the maximum and minimum values of $f(x, y) = 2xy - x - y$ on the square region whose vertices are (0,0), (1,0), (1,1) and (0,1).
- Find the minimum value of $f(x, y) = x^2 - x + y - xy$ on the triangular region whose vertices are (0,0), (2,0) and (2,4).

(8) LAGRANGE MULTIPLIERS

- Find all the extrema of $f(x, y) = x + y$ subject to the constraint that $g(x, y) = x^3 + y^3 = 16$.
- Find all the extrema of $f(x, y) = x^2 + y^2$ subject to the constraint that $g(x, y) = x^4 + y^4 = 16$.
- Find the maxima and minima of $f(x, y) = x + 2y$ on $x^2 + y^2 \leq 5$ (Note: no critical points)
- The only critical point of $f(x, y) = x^2 + xy + y^2$ is (0,0). Find the maximum of f on the region $x^2 + y^2 \leq 1$.
- Compute the minimum distance from $xy = 4$ to the origin.
- Compute the minimum distance from $x + 2y - 3z = 1$ to (0,0,0).
- Find the minimum value of $f(x, y, z) = x + y - z$ on the sphere $x^2 + y^2 + z^2 = 1$.
- A box has a square base, 4 sides but no top. The material costs for the base are twice that of the sides. What relative dimensions minimize the material cost?
- Design a cylindrical can without a top which holds 1 liter of fluid which is made of minimal material (surface area).

(1) PARTIAL DERIVATIVES AND IMPLICIT DIFF.

(a) $f_x(x, y) = \frac{y}{(x+y)^2}$ $f_x(2, -1) = -1$

$f_{xx}(x, y) = \frac{-2y}{(x+y)^3}$ $f_{xx}(2, -1) = 2$

(b) $f_{xz}(x, y, z) = 0$ $f_{xz}(1, 3, 1) = 0$

$f_{yz}(x, y, z) = 2z$ $f_{yz}(1, 3, 1) = 2$

(c) $f_x = \frac{1}{\sqrt{2x+3y}}$ $f_x(1, 1) = \frac{1}{\sqrt{5}}$

$f_y = \frac{3}{2\sqrt{2x+3y}}$ $f_y(1, 1) = \frac{3}{2\sqrt{5}}$

(d) $f_x = e^{(x-1)}(\sin xy + y \cos xy)$, $f_x(1, 0) = 0$

(e) Note that $(x, y) = (1, 1) \Rightarrow z^3 - z = 0$
so that $z(1, 1) = 0, \pm 1$. Implicit
differentiation yields

$$z_x(1, 1) = \frac{\partial x z^3 + 1}{y - 3x^2 z^2} \Big|_{(x, y, z) = (1, 1, \pm 1)} = \frac{1}{2}, -\frac{3}{2}$$

(f) Implicit Diff in x

$$2x + 4z^3 z_x - z_x = 0 \quad z_x = \frac{2x}{(1-4z^3)}$$

Differentiation eqn above in x again.

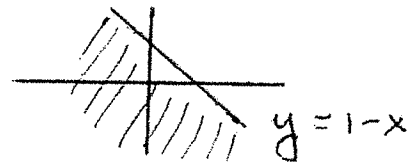
$$2 + 12z^2 z_x^2 + (4z^3 - 1) z_{xx} = 0$$

and solve for z_{xx}

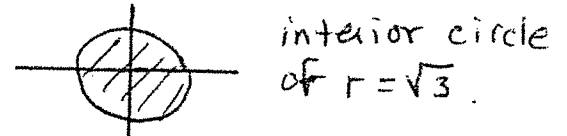
$$z_{xx} = \frac{2 + 12z^2 z_x^2}{(1-4z^3)} = -2 \frac{(16z^6 - 8z^3 + 1 + 24x^2 z^2)}{(4z^3 - 1)^3}$$

(2) LIMITS AND DOMAINS

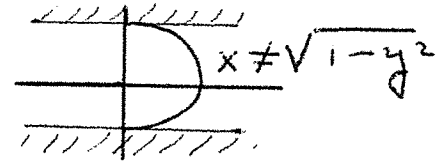
(a) (i) $D = \{(x, y) : 1 - x - y \geq 0\}$



(ii) $D = \{(x, y) : x^2 + y^2 < 3\}$



(iii) $D = \{(x, y) : y^2 \leq 1, x \neq \sqrt{1 - y^2}\}$



(b) Since the limits exist, can evaluate on any path.

(i) $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{x \rightarrow 0} f(x,0) = \lim_{x \rightarrow 0} \frac{x^3}{x+x^2} = 0$

(ii) $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{x \rightarrow 0} f(x,0) = \lim_{x \rightarrow 0} \frac{x^2 - x}{-3x} = \frac{1}{3}$

(iii) $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{x \rightarrow 0} f(x,x) = \lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^2} = 1$

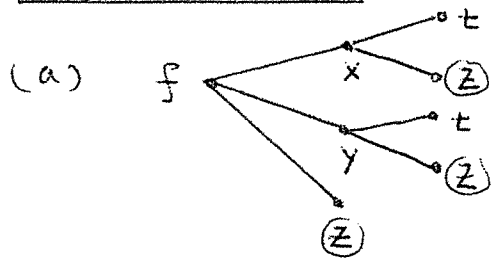
(c) Need to show limits on 2 paths have diff. values.

(i) $\lim_{x \rightarrow 0} f(x, kx) = \frac{k}{2 + 3k^2}$ diff for diff k vals.

(ii) $\lim_{y \rightarrow 2} f(1, y) = \lim_{y \rightarrow 2} \frac{(-2+y)}{(2-y)} = -1$
 $\lim_{x \rightarrow 1} f(x, 2) = \lim_{x \rightarrow 1} \frac{x-1}{x-1} = +1$ } different!

(iii) $\lim_{x \rightarrow 0} f(x, kx^2) = \frac{k}{1+k^2}$ different for diff $y = kx^2$ paths.

(3) CHAIN RULE



$$F(z, t) = f(x, y, z)$$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial f}{\partial z}$$

$$\frac{\partial F}{\partial z} = \partial x \cdot 1 + \partial y \cdot (-3) + 3$$

and at $(z, t) = (1, 1)$ we have $x = 3, y = -2$ so
 $F_z(1, 1) = 21$. Likewise $F_t = \partial x \frac{dx}{dt} + \partial y \frac{dy}{dt} \Big|_{(1, 1)} = 8$.

(b)
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} = (2x - y) \cdot 1 - x(1)$$

evaluate at $(x, y) = (1, 0)$ yields $w_u =$.

(c)
$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = (x^2 - y)(2t + 1) + (x + y)(-3)$$

At $t = 1$ one has $x = 2, y = -2$ thus we evaluate the above expression at $(t, x, y) = (1, 2, -2)$:

$$\left. \frac{df}{dt} \right|_{t=1} = (2^2 + 2)(3) + (2 - 2)(-3) = 18$$

(4) GRADIENT AND DIRECTIONAL DERIVATIVES

(a)
$$\vec{\nabla} f = \left(2x + \frac{1}{2\sqrt{y-x}}, -\frac{1}{2\sqrt{y-x}} \right) \quad \vec{\nabla} f(2, 1) = \left\langle \frac{5}{2}, -\frac{1}{2} \right\rangle$$

(b) First $\vec{\nabla} f = \langle 3x^2y, x^3 - 1 \rangle, \vec{\nabla} f(1, 1) = \langle 3, 0 \rangle$.
Direction $\vec{u} = \langle 1, 3 \rangle$, unit direction $\hat{u} = \frac{1}{\sqrt{10}} \langle 1, 3 \rangle$

$$D_{\hat{u}} f(1, 1) = \vec{\nabla} f(1, 1) \cdot \hat{u} = \frac{3}{\sqrt{10}}$$

Direction of decreasing most rapidly is $-\vec{\nabla} f(1, 1) = \langle -3, 0 \rangle$.

c) Directional derivative from (1,1) "toward" (4,5).

$$\vec{\nabla} f = \langle y, x - 2y \rangle \quad \vec{\nabla} f(1,1) = \langle 1, -1 \rangle$$

direction vector $Q(4,5) - P(1,1) = \langle 3, 4 \rangle = \vec{u} \Rightarrow \hat{u} = \frac{1}{5} \langle 3, 4 \rangle$
is unit direction vector.

$$D_{\hat{u}} f(1,1) = \vec{\nabla} f(1,1) \cdot \hat{u} = \frac{1}{5} (3 - 4) = -\frac{1}{5}$$

d) For speed $v = 2 \text{ ft/sec}$ the answer is $v \cdot D_{\hat{u}} f(1,1)$
where $\hat{u} = \langle 0, 1 \rangle$ for positive y direction
along $x=1$.

$$\text{ANS} = v \vec{\nabla} f(1,1) \cdot \langle 0, 1 \rangle = v f_y(1,1) = v \left(\frac{-200xy}{(x^2+y^2)^2} \right) \Big|_{(1,1)} = -100 \frac{\text{ft}}{\text{sec}}$$

e) For $f(x,y) = F(r)$

$$f_x = F'(r) \frac{\partial}{\partial x} \sqrt{x^2+y^2} = F'(r) \frac{x}{\sqrt{x^2+y^2}} = F'(r) \frac{x}{r}$$

Similarly, $f_y = F'(r) \cdot \frac{y}{r}$ so that

$$\vec{\nabla} f = \langle f_x, f_y \rangle = F'(r) \left\langle \frac{x}{r}, \frac{y}{r} \right\rangle = F'(r) \cdot \frac{1}{r} \vec{r}$$

By defn the unit radial vector $\hat{r} = \frac{1}{r} \vec{r}$ so

$$\vec{\nabla} f = F'(r) \hat{r}$$

(5) GEOMETRY

a) $\vec{N} = \langle -f_x(1,1), -f_y(1,1), 1 \rangle = \langle -2, -3, 1 \rangle$ $|\vec{N}| = \sqrt{14}$

Two unit vectors are $\pm \hat{N} = \pm \frac{1}{\sqrt{14}} \langle -2, -3, 1 \rangle$

b) $\vec{T}_1 = \langle f_x(1,1), 0, 1 \rangle = \langle 1, 0, 1 \rangle$

$\vec{T}_2 = \langle 0, f_y(1,1), 1 \rangle = \langle 0, 3, 1 \rangle$

c) Tangent Plane

$$f(x,y) = x + y + xy^3$$

$$f(1,1) = 3$$

$$f_x(x,y) = 1 + y^3$$

$$f_x(1,1) = 2$$

$$f_y(x,y) = 1 + 3xy^2$$

$$f_y(1,1) = 4$$

Normal $\vec{N} = \langle -2, -4, 1 \rangle$ and $\vec{r}_0 = \langle 1, 1, 3 \rangle$ on plane

$$2x + 4y - z = 3$$

d) Normal $\vec{N} = \langle -f_x(1,1), -f_y(1,1), 1 \rangle = \langle -\frac{1}{4}, -\frac{3}{4}, 1 \rangle$

Thru $\vec{r}_0 = \langle x_0, y_0, f(x_0, y_0) \rangle = \langle 1, 1, 2 \rangle$ yields

$$\vec{r}(t) = \langle 1, 1, 2 \rangle + t \langle -\frac{1}{4}, -\frac{3}{4}, 1 \rangle \quad \text{Normal Line.}$$

e) Normal Vector $\vec{N} = \langle -f_x(1,1), -f_y(1,1), 1 \rangle = \langle -\frac{1}{6}, -\frac{1}{8}, 1 \rangle$

Thru $\vec{r}_0 = \langle 1, 1, f(1,1) \rangle = \langle 1, 1, \frac{1}{4} \rangle$ yields

$$\vec{r}(t) = \langle 1 - \frac{1}{6}t, 1 - \frac{1}{8}t, \frac{1}{4} + t \rangle \quad \text{Normal Line L}$$

intersects xy-plane when $z(t) = \frac{1}{4} + t = 0 \Leftrightarrow t = -\frac{1}{4}$

Intersection P: $\vec{r}(-\frac{1}{4}) = \langle \frac{65}{64}, \frac{33}{32}, 0 \rangle$

$$f) \quad \vec{\nabla} f(x, y) = \langle 2xy, x^2 \rangle \quad \vec{\nabla} f(1, 1) = \langle 2, 1 \rangle \perp \text{ curve } x^2y = 1 \text{ @ } (1, 1).$$

$$g) \quad \text{Level surface } g(x, y, z) = x^2 - z^3 + yz = 0$$

$$\vec{\nabla} g(x, y, z) = \langle 2x, z, y - 3z^2 \rangle \perp \text{ surface}$$

When $(x, y) = (1, 0)$, $g(1, 0, z) = 1 - z^3 = 0 \Leftrightarrow z = 1$
 so $(x, y, z) = (1, 0, 1)$ on surface

$$\vec{\nabla} g(1, 0, 1) = \langle 2, 1, -3 \rangle \perp \text{ surface @ } (1, 0, 1).$$

$$h) \quad \vec{\nabla} f(x, y) = \langle 2x, 3y^2 \rangle. \quad \vec{\nabla} f(1, 1) = \langle 2, 3 \rangle \text{ is } \perp \text{ at } (1, 1) \text{ but not unit length.}$$

$$|\vec{\nabla} f(1, 1)| = \sqrt{13} \quad \hat{N} = \frac{1}{\sqrt{13}} \langle 2, 3 \rangle \perp$$

i) Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ be the curve of intersection. Since $z = f(x, y)$ on such a curve, 'one' parametrization is $(x(t)^2 + y(t)^2 = 1)$

$$\vec{r}(t) = \langle \cos t, \sin t, f(x(t), y(t)) \rangle$$

$$\vec{r}'(t) = \langle -\sin t, \cos t, f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} \rangle$$

$$\vec{r}'(t) = \langle -\sin t, \cos t, (x+y)(-\sin t) + (x-y)\cos t \rangle$$

when $(x, y) = (1, 0)$, $t = 0$ so tangent vector

$$\vec{r}'(0) = \langle 0, 1, 1 \rangle$$

by evaluating $\vec{r}'(t)$ above.

(6) CRITICAL POINTS AND 2ND DERIVATIVE TEST

a(i) $f = x^2 + 2y^2 - 4x + 4y$

$$\left. \begin{aligned} f_x &= 2x - 4 = 0 \\ f_y &= 4y + 4 = 0 \end{aligned} \right\} \begin{aligned} (x,y) &= (2,-1) \\ &\text{sole ct. pt.} \end{aligned}$$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (2)(4) - 0^2 = 8 > 0 \text{ and } f_{xx} = 2 > 0 \text{ hence } (2,-1) \text{ is a local min.}$$

a(iv) $f = x^4 + y^4 - 4xy$

(1) $f_x = 4x^3 - 4y = 0$

(2) $f_y = 4y^3 - 4x = 0$

Eqn true only if $y = x^3$. Using this in (2)

$$x^9 - x = x(x^8 - 1) = 0 \quad x = 0, \pm 1$$

Yields 3 ct. pts. $(0,0)$ $(1,1)$ and $(-1,-1)$

(x,y)	$D = 144x^2y^2 - 16$	$f_{xx} = 12x^2$	Conclude
$(0,0)$	-	+	saddle
$(1,1)$	+	+	loc. min
$(-1,-1)$	+	+	loc. min

b) Since $\nabla f = \langle x^2 - y, y - x \rangle$ then

(1) $f_x = x^2 - y = 0$

(2) $f_y = y - x = 0$

yields $x^2 - x = x(x-1) = 0$ so ct pts $(0,0)$ and $(1,1)$.

From (1), $f_{xx} = 2x$, $f_{xy} = -1$. From (2), $f_{yy} = 1$.

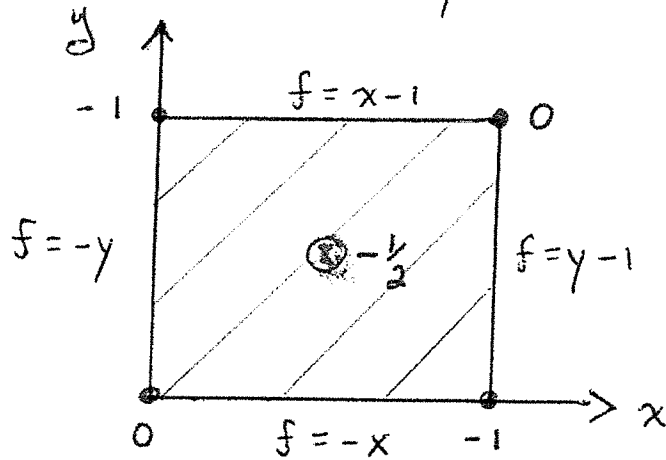
Conclude $D = 2x - 1$ and classifying

$(0,0)$ saddle

$(1,1)$ local min

(7) MAXIMA AND MINIMA ON A REGION

a) The function $f = 2xy - x - y$ has the sole critical point $(x_0, y_0) = (\frac{1}{2}, \frac{1}{2})$ and it is inside the unit square.



⊗ ct pt

$$f(0, y) = -y$$

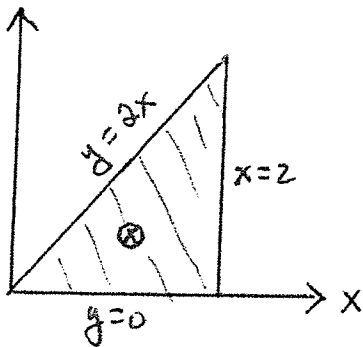
$$f(x, 1) = x - 1$$

$$f(1, y) = y - 1$$

$$f(x, 0) = -x$$

Since $f(\frac{1}{2}, \frac{1}{2}) = -\frac{1}{2}$ the max f is 0 and it occurs at $(0, 0), (1, 1)$ on boundary.

b) $f = x^2 - x + y - xy$ has one ct. pt. at $(x, y) = (1, 1)$.



$$f(x, 0) = x^2 - x = (x - \frac{1}{2})^2 - \frac{1}{4}$$

$$f(2, y) = 2 - y$$

$$f(x, 2x) = x - x^2 = -(x - \frac{1}{2})^2 + \frac{1}{4}$$

are values of f on 3 edges

$$\max_{x \in [0, 2]} f(x, 0) = f(2, 0) = 2$$

$$\max_{y \in [0, 4]} f(2, y) = f(2, 0) = 2$$

$$\max_{x \in [0, 2]} f(x, 2x) = f(\frac{1}{2}, 1) = \frac{1}{4}$$

$$\min_{x \in [0, 2]} f(x, 0) = f(\frac{1}{2}, 0) = -\frac{1}{4}$$

$$\min_{y \in [0, 4]} f(2, y) = f(2, 4) = (-2)$$

$$\min_{x \in [0, 2]} f(x, 2x) = f(2, 4) = (-2)$$

Given, at ct. pt, $f(1, 1) = 0$ then abs min/max:

$$\text{abs max} = +2 \quad \text{at } (2, 0) \quad]$$

$$\text{abs min} = -2 \quad \text{at } (2, 4) \quad \circ$$

(8) LAGRANGE MULTIPLIERS

$$\begin{array}{ll} \text{a)} & \begin{array}{l} (1) \quad 1 = 3x^2\lambda \\ (2) \quad 1 = 3y^2\lambda \\ (3) \quad x^3 + y^3 = 16 \end{array} \end{array} \quad \begin{array}{l} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{array}$$

Divide (2) and (1) since $x, y, \lambda \neq 0$. $y^2 = x^2$
hence $y = \pm x$.

$y = -x$ in (3) yields the contradiction $0 = 16$.
Thus $y = +x$ only which in (3) yields $2x^3 = 16$
or $x = 2$.

$$(x, y) = (2, 2) \quad \text{sole extrema.}$$

b) Problem has many many extrema.

$$\begin{array}{ll} (1) & 2x = 4\lambda x^3 \\ (2) & 2y = 4\lambda y^3 \\ (3) & x^4 + y^4 = 16 \end{array}$$

$$x = 0 \text{ in (1) and (3)} \Rightarrow y = \pm 2 \quad (x, y, \lambda) = (0, \pm 2, \frac{1}{8})$$

$$y = 0 \text{ in (2) and (3)} \Rightarrow x = \pm 2 \quad (x, y, \lambda) = (\pm 2, 0, \frac{1}{8})$$

If both $x, y \neq 0$ then dividing (2) and (1) implies
 $y = \pm x$. Using $y = \pm x$ in (3) one finds

$$2x^4 = 16 \quad x = \pm \sqrt[4]{8}$$

Thus we have (in totality) 8 extrema

$$(x, y) = (0, \pm 2), (\pm 2, 0)$$

$$(x, y) = \pm (\sqrt[4]{8}, \sqrt[4]{8}), \pm (\sqrt[4]{8}, -\sqrt[4]{8})$$

c) Since f has no critical points the max/min must occur on the boundary.
Must solve (for $f = x + 2y$)

$$\max f(x, y)$$

$$g(x, y) = x^2 + y^2 = 5$$

Lagrange Multiplier λ .

$$(1) \quad 1 = 2\lambda x$$

$$(2) \quad 2 = 2\lambda y$$

$$(3) \quad x^2 + y^2 = 5$$

Can't have $x, y = 0$ in (1) or (2) so both $x, y \neq 0$.
Thus divide (1) and (2)

$$y = 2x$$

which used in (3) yields $5x^2 = 5$ or $x = \pm 1$

$$(x, y) = (1, 2)$$

$$(x, y) = (-1, -2)$$

$$f = 5$$

$$f = -5$$

abs max

abs min

d) Must find extrema on boundary and compare to value $f(0, 0) = 0$ at sole critical point $(0, 0)$ inside region

$$(1) \quad 2x + y = 2\lambda x$$

$$(2) \quad x + 2y = 2\lambda y$$

$$(3) \quad x^2 + y^2 = 1$$

} divide and simplify
to get $y^2 = x^2$

Eqns (1)-(2) imply $y = \pm x$ which used in (3) yields $2x^2 = 1$, $x = \pm \frac{1}{\sqrt{2}}$ or 4 points

$$f = \begin{matrix} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ \text{abs max} & & & \text{abs max} \end{matrix}$$

C + P + is abs min since $f(0, 0) = 0$ smallest.

e) Sufficient to minimize distance squared $f = x^2 + y^2$ subject to constraint $g = xy - 4 = 0$

$$\begin{aligned}(1) \quad 2x &= \lambda y \\(2) \quad 2y &= \lambda x \\(3) \quad xy &= 4\end{aligned}$$

Clearly neither $x, y = 0$ else (3) not satisfied. Thus divide (1) and (2) to get (after simplifying)

$$y = \pm x \quad (y^2 = x^2)$$

which used in (3) yields $x^2 = 4$, $x = \pm 2$

The minimum distance is $\sqrt{2^2 + 2^2} = \sqrt{8}$ and occurs at $\pm (2, 2)$ points.

f) Minimize $f = x^2 + y^2 + z^2$ subject $g = x + 2y - 3z = 1$

$$\begin{aligned}(1) \quad 2x &= \lambda & \Rightarrow & x = \frac{\lambda}{2} \\(2) \quad 2y &= 2\lambda & \Rightarrow & y = \lambda \\(3) \quad 2z &= -3\lambda & \Rightarrow & z = -\frac{3}{2}\lambda \\(4) \quad x + 2y - 3z &= 1\end{aligned}$$

using $x = \frac{1}{2}\lambda$, $y = \lambda$, $z = -\frac{3}{2}\lambda$ in Eqn (4) yields

$$7\lambda = 1$$

hence $\lambda = \frac{1}{7}$ and

$$(x, y, z) = \left(\frac{1}{14}, \frac{1}{7}, -\frac{3}{14}\right)$$

is the point on the plane closest to $(0, 0, 0)$. The minimum distance is

$$D_{\min} = \sqrt{\left(\frac{1}{14}\right)^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{3}{14}\right)^2} = \frac{1}{\sqrt{14}}$$

g) Minimize $f = x + y - z$ sub to $g = x^2 + y^2 + z^2 = 1$

$$\left. \begin{array}{l} (1) \quad 1 = 2\lambda x \\ (2) \quad 1 = 2\lambda y \\ (3) \quad -1 = 2\lambda z \\ (4) \quad x^2 + y^2 + z^2 = 1 \end{array} \right\} \text{solve } x, y, z \text{ in terms of } \lambda.$$

Find $x = \frac{1}{2\lambda}$, $y = \frac{1}{2\lambda}$ and $z = -\frac{1}{2\lambda}$. Use in (4) to find

$$\lambda = \pm \frac{\sqrt{3}}{2} \quad (\lambda^2 = \frac{3}{4})$$

Thus obtain two points (extrema)

$$(x, y, z) = \pm \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) = \pm \vec{X}$$

Evaluating at these pts "+" makes f a max hence min is $f(-\vec{X}) = -\sqrt{3}$ at $-\vec{X}$.

h) Base x , height y . Cost $C(x, y) = 2x^2 + 4xy$.
Minimize cost $C(x, y)$ subject to fixed volume $g = x^2 y = V$.

$$\left. \begin{array}{l} C_x = \lambda g_x \\ C_y = \lambda g_y \end{array} \right\} \text{divide yields } y = x, \text{ i.e. base the same as height (or a cube!)}$$

i) Radius r , height h . Note liter = 10^3 cm^3 .

$$\begin{array}{l} S = 2\pi r h + \pi r^2 \quad (\text{surface area}) \\ V = \pi r^2 h \quad (\text{Volume}) \end{array}$$

Minimize S subject to $V = 1$

$$\left. \begin{array}{l} (1) \quad S_r = \lambda V_r \\ (2) \quad S_h = \lambda V_h \\ (3) \quad V = 1000 \text{ cm}^3 \end{array} \right\} r = h = \frac{10}{\sqrt[3]{\pi}} \text{ cm}$$