Review Questions

(I) DOUBLE INTEGRALS - Cartesian and Polar Coordinates

1. Compute $\int\int_R dA$ where R is the region bounded by $y = 3 - x$, and the x and y axes.

2. Evaluate $\int\int_R (x - y)dA$ where R is the triangle with vertices $(0,0), (1,0)$ and $(2,1)$.

3. Interchange the limits of integration and then evaluate:

$$a) \quad \int_0^1 \int_y^1 e^{-x^2} dx dy \qquad b) \quad \int_0^{\pi/2} \int_y^{\pi/2} \frac{sin(x)}{x} dx dy \qquad (1)$$

4. Use polar coordinates to evaluate $\int\int_R (x+y)dA$ on the region in the first quadrant under $y = \sqrt{3}x$ and inside $x^2 + y^2 = 9$.

5. Set up the integral $\int\int_R (x^2 + y^2)dA$ in polar coordinates where R is the triangle with vertices $(0,0), (1,0)$ and $(1,1)$.

6. Re-express the following iterated integral in polar coordinates:

$$\int_0^2 \int_0^{\sqrt{4-x^2}} xy \; dy \; dx \qquad (2)$$

7. Let R be the region in the first quadrant bounded by the y axis, the line $y = x$ and the circle $x^2 + y^2 = 4$. Draw R and then evaluate $\int\int_R \sqrt{x^2 + y^2} dA$.

8. Evaluate $\int\int_R x \; dA$ where R is the region inside the circle $x^2 + y^2 = 8$ with $y > x^2/2$.

(II) VOLUME INTEGRALS - Cartesian, Cylindrical(C), Spherical(S) Coordinates

1. Set up the cartesian and cylindrical integrals whose values are the volume of the region bounded by $z = x^2 + y^2$ and $3z = 4 - x^2 - y^2$.

2. Set up the cartesian and cylindrical integrals whose values are the volume inside the cylinder $x^2 + y^2 = 4$, below the plane $z = x + y + 4$ and above $z = 0$.

3. Evaluate $\int\int\int_R (x^2 + y^2)dV$ on the cube $0 < x, y, z < 1$.

4. Evaluate $\int\int\int_R 3dV$ on the upper hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$.

5. Find the volume inside $x^2 + y^2 = 4$, above $z = 0$, below $z = x + 2$ and having $x > 0$.

6. Set up the iterated integral for $\int\int\int_R y dV$ where R is the portion of the cube $0 < x, y, z < 1$ above $y + z = 1$ and below $x + y + z = 2$. (Project onto yz-plane).

7. For each of the following regions, setup an integral for the volume

a) (C) Above $z = (x^2 + y^2)^{1/4}$ and inside $x^2 + y^2 + z^2 = 1$.

b) (C) Between the paraboloids $z = 10 - x^2 - y^2$ and $z = 2(x^2 + y^2 - 1)$.

c) (C) Above the xy-plane, under $z = 1 - x^2 - y^2$ and inside the wedge $-x \leq y \leq \sqrt{3}x$.

d) (S) The interior of a sphere of radius 2 (center origin) less the portion with $z > \sqrt{x^2 + y^2}$

8. Find the mass of the upper hemisphere of radius 5 whose density is $f(x, y, z) = 10 - z$.

9. Let R be the region inside the cylinder $x^2 + y^2 = 1$, above the xy plane and below $z = y + 1$. The density of the solid is $F(x, y, z) = x^2 + y^2 + z^2$. Set up an iterated integral in cylindrical coordinates for the mass of the solid.

10. Use spherical coordinates to setup an iterated integral whose value is the mass of a solid R above $z = 0$, bounded by the unit sphere $x^2 + y^2 + z^2 = 1$ and the cones $z^2 = x^2 + y^2$, $z^2 = 3(x^2 + y^2)$. The density of the solid is $f(x, y, z) = z^2 + 1$ (Note: $z = \rho cos\phi$ in spherical coordinates).

(III) LINE INTEGRALS

1. Let $\vec{F} = (x, x, z)$. Evaluate $\int_C \vec{F} \cdot \vec{dr}$ on the curves

(a) straight line from (0,0,0) to (1,1,1).

(b) the curve parametrized by $\vec{r}(t) = (t, t^2, t^3), 0 \leq t \leq 1$.

2. Compute $\int_C x^2 dx$ on the unit circle (counterclockwise).

4. Evaluate the line integral

$$\int_C y^2 dx \tag{3}$$

where C is the straight line from $(2, 1, 0)$ to $(1, 3, 7)$.

5. Compute the work done by $\vec{F} = (x^2 + xy, y - x^2 y)$ along the path C parametrized by $\vec{r}(t) = (t, 1/t), 1 < t < 3$.

6. Evaluate

$$\int_C (y - x)dx + xydy$$

where C is

a) The straight line from $(1, 1)$ to $(2, 3)$.

h) Is the portion of the parabola $y = x^2, 0 < x < 1$ directed in the positive x direction.

(IV) SURFACE INTEGRALS

1. Compute the surface area of the portion of the plane $x + 2y + z = 4$ inside the cylinder $x^2 + y^2 = 1$.

2. Compute the surface area of the paraboloid $z = 4 - x^2 - y^2$ above the $z = 1$ plane.

3. Set up a polar integral whose value is $\int\int_S z dS$ where S is the portion of the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 4$.

4. Evaluate $\int\int_S z^2 dS$ on the upper portion of the sphere $x^2 + y^2 + z^2 = 1$.

5. Set up an iterated integral whose value is $\int\int_S \vec{r} \cdot \vec{r} dS$, $\vec{r} = (x, y, z)$ and S being the plane $z = 1$ over the unit square $0 \leq x, y \leq 1$.

6. Find the surface area of the portion of the paraboloid $z = 2 - x^2 - y^2$ that is above the plane $z = 1$ and has $y \leq x$.

7. A surface S is the portion of the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 4$. The surface has density $\rho(x, y, z) = x^2 z$ (kg/m^2). Compute the mass.

(V) FLUX AND OTHER SURFACE INTEGRALS

1. Compute the flux of $\vec{F} = z\hat{i}$ through the upper unit hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ where the unit normal $\hat{N}$ is oriented in the positive z direction.

2. Compute the flux of $\vec{F} = z\hat{k}$ through the portion of $x^2 + y^2 + z^2 = 4$ in the positive octant $(x, y, z > 0)$ where the unit normal $\hat{N}$ is oriented upward ($+z$ direction).

3. Compute the flux of $\vec{F} = (x, y, z)$ through the portion of the parabolic sheet $z = x^2$, $x \in (0, 4)$, $y \in (0, 1)$ with $\hat{N}$ oriented upward ($+z$ direction).

4. Compute $\int_S \vec{F} \cdot \hat{N} dS$ where $\vec{F} = (z - x, 0, z + 1)$ and S is the portion of the plane $2x + y + z = 1$ in the positive octant and $\hat{N}$ is oriented upwards.

(VI) CONSERVATIVE FIELDS

1. Use the curl to determine which of the following vector fields is conservative.

(i) $\vec{F}(x, y) = (xy - y, x - x^2)$

(ii) $\vec{F}(x,y) = (y^2 - ye^{xy}, 2xy - xe^{xy})$

(iii) $\vec{F}(x,y,z) = (xy, xy, xz)$

2 Find the associated potential function for each of the following vector fields

(i) $\vec{F}(x,y) = (3x^2y + y, x^3 + x - 2y)$

(ii) $\vec{F}(x,y) = (ye^{xy} + \frac{1}{x+2y}, xe^{xy} + \frac{2}{x+2y})$

(iii) $\vec{F}(x,y,z) = (2xy, x^2 - z, 1 - y)$

3 Evaluate the following line integrals

(i) $\int_{(0,1)}^{(2,2)} (1+y)dx + (x+1)dy$

(ii) $\int_C \frac{1}{y}dx + \left(1 - \frac{x}{y^2}\right) dy$ where C is parametrized by $\vec{r}(t) = (t+1, 1+\sqrt{2t^2+1}), 0 < t < 2$

(iii) $\int_C dx + dy + 2zdz$ where C is any simple curve from the origin to $(1,1,1)$.

(VII) DIVERGENCE THEOREM

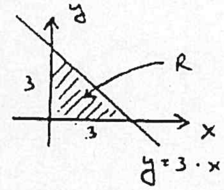
1 Let V be the solid with $\sqrt{x^2+y^2} < z < 1$ and S its bounding surface. Compute the flux of $\vec{F} = (y^2x + z, y + 1, zx^2 - y)$ out of S.

2 Let V be the solid with $0 < z < 1 - y^2, 0 < x < 1$ and S its bounding surface. Compute the flux of $\vec{F} = (x,y,z)$ out of S.

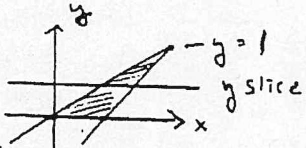
3 Use the divergence theorem to compute the flux of $\vec{F} = (y - z, yx^2, zy^2)$ through the portion of the paraboloid $z = x^2 + y^2, z < 1$ and the unit normal is oriented in the negative z direction (downward).

SECTION I

1) $\iint_R dA$ = area of triangle $= \frac{9}{2}$



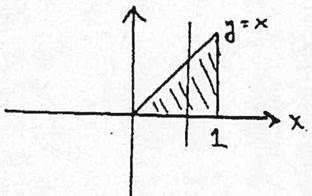
2)



$x = 2y \quad x = y + 1$

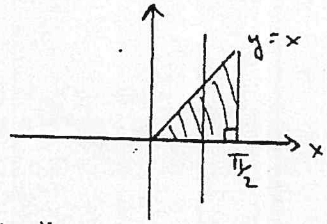
$$\int_0^1 \int_{2y}^{y+1} (x-y)\,dx\,dy = +\frac{1}{3}$$

3) a)



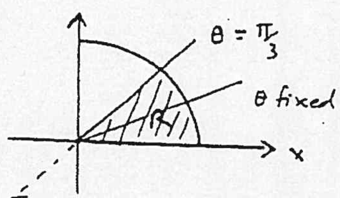
$$\int_0^1 \int_0^x e^{-x^2}\,dy\,dx = \tfrac{1}{2}(1-e^{-1})$$

3) b)



$$\int_0^{\pi/2} \int_0^x \frac{\sin x}{x}\,dy\,dx = 1$$

4)



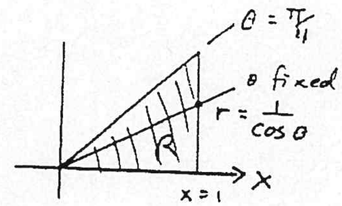
$y = \sqrt{3}x$

$0 < r < 3$
$0 < \theta < \pi/3$

$$\int_0^{\pi/3} \int_0^3 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = \frac{9}{2}(\sqrt{3}+1)$$

1.

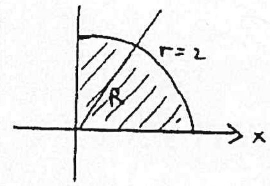
5)



$$\int_0^{\pi/4} \int_0^{\frac{1}{\cos\theta}} r^3\,dr\,d\theta = \frac{1}{3}$$

(HARD INTEGRAL)

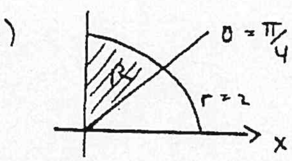
6)



$y = \sqrt{4-x^2} \iff r^2 = 4$

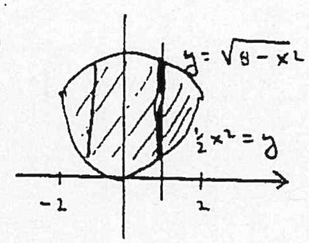
$$\int_0^{\pi/2} \int_0^2 r^3 \cos\theta \sin\theta\,dr\,d\theta = 2$$

7)



$$\int_{\pi/4}^{\pi/2} \int_0^2 r^2\,dr\,d\theta = \frac{8\pi}{3}$$

8)



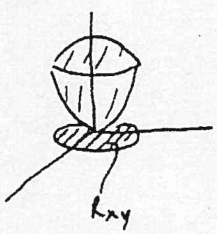
Intersection $x^2 + \frac{1}{4}x^4 = 8 \Rightarrow x = 2$

$$\int_{-2}^{2} \int_{\frac{1}{2}x^2}^{\sqrt{8-x^2}} x\,dy\,dx = 0$$

2.

ECTION II

1)



$z = r^2 \qquad z = \frac{1}{3}(4-r^2)$

Intersection $r^2 = \frac{1}{3}(4-r^2) \Rightarrow r = 1$

$$\int_0^{2\pi}\int_0^{1}\int_{r^2}^{\frac{1}{3}(4-r^2)} r\,dz\,dr\,d\theta = \frac{2\pi}{3}$$

Cartesian is

$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}}\int_{x^2+y^2}^{\frac{1}{3}(4-x^2-y^2)} dz\,dy\,dx$$

2)
$$V = \int_0^{2\pi}\int_0^{2}\int_0^{4+r\cos\theta+r\sin\theta} r\,dz\,dr\,d\theta = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{x+y+4} dz\,dy\,dx$$

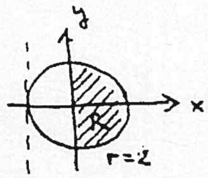
3)
$$\int_0^1\int_0^1\int_0^1 (x^2+y^2)\,dz\,dy\,dx = \frac{2}{3}$$

4)
$$\iiint_R 3\,dV = 3\left(\begin{matrix}\text{volume of hemisphere}\\ \text{of radius } 4\end{matrix}\right) = 2\pi(4)^3$$

In other coordinates

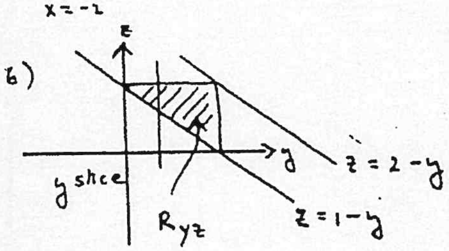
$$3\int_0^{2\pi}\int_0^{4}\int_0^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta = 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^{4} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

5)



$$\int_{-\pi/2}^{\pi/2}\int_0^{2}\int_0^{r\cos\theta+2} r\,dz\,dr\,d\theta = \frac{16}{3} + 4\pi$$

6)



Project onto yz-plane

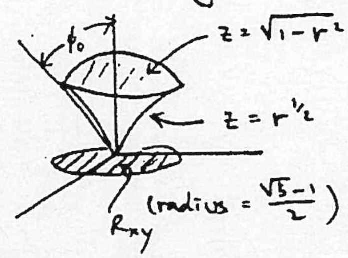
$x = 0$
$x = 2 - y - z$

On R_{yz} have

$0 < 2 - y - z < 1$

$$\int_0^1\int_{1-y}^{1}\int_0^{2-y-z} y\,dx\,dz\,dy = \frac{5}{24}$$

7 a) $z = (x^2+y^2)^{1/4} \Rightarrow z = r^{1/2}$ Intersect



$r = 1 - r^2$

$r^2 + r - 1 = 0 \quad \Rightarrow r = \frac{\sqrt{5}-1}{2}$

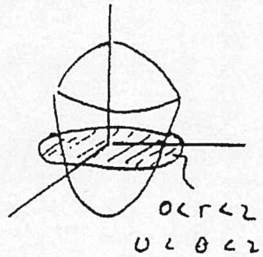
$$V = \int_0^{2\pi}\int_0^{\frac{\sqrt{5}-1}{2}}\int_{r^{1/2}}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta$$

In spherical:

$$V = \int_0^{2\pi}\int_0^{\phi_0}\int_{\frac{\sin\phi}{\cos^2\phi}}^{1} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta,$$

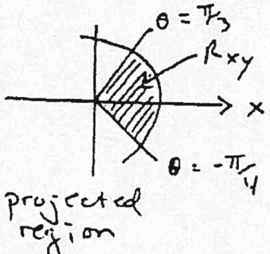
$$\phi_0 = \arcsin\left(\frac{\sqrt{5}-1}{2}\right)$$

7b) $z = 10 - r^2$, $z = 2r^2 - 2$ intersect at $r = 2$



$$V = \int_0^{2\pi}\int_0^{2}\int_{2r^2-2}^{10-r^2} r\,dz\,dr\,d\theta = 24\pi$$

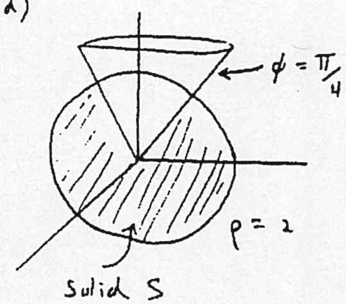
7c) Have $0 < z < 1 - r^2$ (Intersect $r = 1$)



$$V = \int_{-\pi/4}^{\pi/3}\int_0^{1}\int_0^{1-r^2} r\,dz\,dr\,d\theta$$

$$= \frac{7\pi}{48}$$

7d)



$$V = \int_0^{2\pi}\int_{\pi/4}^{\pi}\int_0^{2} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \frac{8\pi}{3}(2+\sqrt{2})$$

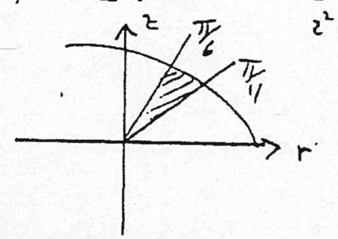
8) $$M = \iiint_V f\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{5} (10 - \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

In cylindrical coordinates it would be

$$M = \int_0^{2\pi}\int_0^{5}\int_0^{\sqrt{25-r^2}} (10 - z)\,r\,dz\,dr\,d\theta = \frac{8125\pi}{12}$$
(HARD)

9) $$M = \int_0^{2\pi}\int_0^{1}\int_0^{r\sin\theta+1} (r^2+z^2)\,r\,dz\,dr\,d\theta = \frac{13\pi}{12}$$
(HARD)

10) Side view:



$z^2 = x^2 + y^2 \Rightarrow \phi = \pi/4$
$z^2 = 3(x^2+y^2) \Rightarrow \phi = \pi/6$

$$M = \int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_0^{1} (\rho^2\cos^2\phi + 1)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

<u>SECTION III</u>

1a) $\vec{r}(t) = (t, t, t)$, $t \in (0,1)$, $\vec{r}\,'(t) = (1,1,1)$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (t,t,t)\cdot(1,1,1)\,dt = \frac{3}{2}$$

1b) $\vec{r}(t) = (t, t^2, t^3)$, $t \in (0,1)$, $\vec{r}\,'(t) = (1, 2t, 3t^2)$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (t, t, t^3)\cdot(1, 2t, 3t^2)\,dt = \frac{5}{3}$$

2) $\vec{r}(t) = (\cos t, \sin t)$, $t \in (0, 2\pi)$, $\vec{F} = (x^2, 0)$, $\vec{r}\,' = (-\sin t, \cos t)$

$$\int_C x^2 dx = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (\cos^2 t, 0)\cdot(-\sin t, \cos t)\, dt = 0$$

3) $\vec{r}(t) = (t, t^2)$, $t \in (-1, 1)$, $\vec{F} = (x, y)$, $\vec{r}\,' = (1, 2t)$

$$\int_C x\,dx + y\,dy = \int_{-1}^{1} (t, t^2)\cdot(1, 2t)\, dt = 0$$

4) $P = (2, 1, 0)$, $Q = (1, 3, 7)$, $\vec{r}(t) = \vec{OP} + t\,\vec{PQ}$, $t \in (0, 1)$

$\vec{r}(t) = (2-t, 1+2t, 7t)$ $\quad \vec{r}\,'(t) = (-1, 2, 7)$

Also, $\vec{F} = (y^2, 0, 0)$

$$\int_C y^2 dx = \int_0^1 ((1+2t)^2, 0, 0)\cdot(-1, 2, 7)\, dt = -\frac{13}{3}$$

5) $\vec{r}(t) = (t, 1/t)$, $\vec{r}\,'(t) = (1, -\frac{1}{t^2})$, $t \in (1, 3)$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_1^3 \left(t^2+1, \frac{1}{t} - t\right)\cdot\left(1, -1/t^2\right) dt = \frac{92}{9} + \ln 3$$

6a) $\vec{r}(t) = (1+t, 1+2t)$, $\vec{r}\,' = (1, 2)$, $t \in (0, 1)$

$$I = \int_0^1 (t, (1+t)(1+2t))\cdot(1, 2)\, dt = \frac{41}{6}$$

6b) $\vec{r}(t) = (t, t^2)$, $t \in (0, 1)$, $\vec{r}\,' = (1, 2t)$

$$I = \int_0^1 (t^2 - t, t^3)\cdot(1, 2t)\, dt = \frac{7}{30}$$

7) Since $\vec{r}\,'(t) = (1, 6t, 18t^2)$,

$$ds = \|\vec{r}\,'(t)\|\,dt = \sqrt{(18t^2+1)^2} = \sqrt{18^2 t^4 + 6^2 t^2 + 1}$$

Thus

$$\int_C xyz^2\, ds = \int_0^1 108 t^9 (18t^2+1)\, dt = \frac{864}{5}$$

For the second integral, $\vec{F} = (0, 0, xyz^2)$, $\vec{r}\,' = (1, 6t, 18t^2)$

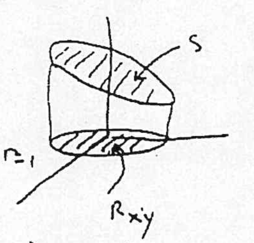
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (0, 0, 108t^9)\cdot(1, 6t, 18t^2)\, dt = 162$$

8) $\vec{r}\,'(t) = (-2\sin t, 2\cos t, 1)$, $\|\vec{r}\,'(t)\| = \sqrt{5}$

$$M = \int_C (x^2+y^2)\, ds = \int_0^{2\pi} (4)\underbrace{\sqrt{5}\, dt}_{ds} = 8\sqrt{5}\,\pi$$

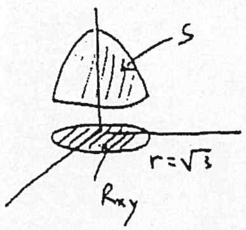
SECTION IV

1) $z = f(x,y) = 4 - x - 2y$, $\|\vec{N}\| = \sqrt{1 + f_x^2 + f_y^2} = \sqrt{6}$



$$\iint_S ds = \sqrt{6} \iint_{R_{xy}} dA = \sqrt{6}\,\pi \cdot 1^2$$

2) $z = f(x,y) = 4 - x^2 - y^2$, $\|\vec{N}\| = \sqrt{1+4r^2}$ in polar.

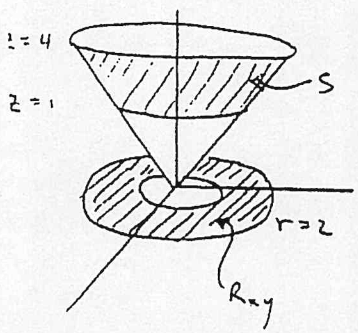


$$\iint_S ds = \int_0^{2\pi}\int_0^{\sqrt{3}} r\sqrt{1+4r^2}\,\underline{dr\,d\theta} = \frac{\pi}{6}(13\sqrt{13}-1)$$

where $\underline{dA} = r\,dr\,d\theta$

3) $z = f(x,y) = \sqrt{x^2+y^2}$, $z_x = \frac{x}{r}$, $z_y = \frac{y}{r}$, $\vec{N} = \left(-\frac{x}{r}, -\frac{y}{r}, 1\right)$

then $\|\vec{N}\| = \sqrt{2}$.

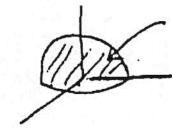


$$\iint_S z\,dS = \iint_{R_{xy}} \sqrt{2}\sqrt{x^2+y^2}\,dA$$

$$= \sqrt{2}\int_0^{2\pi}\int_1^{4} r^2\,dr\,d\theta$$

$$= 42\pi\sqrt{2}$$

4) $z = f(x,y) = \sqrt{1-x^2-y^2}$, $\|\vec{N}\| = \frac{1}{\sqrt{1-r^2}}$ in polar.

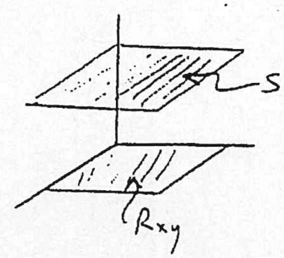


$$\iint_S z^2\,dS = \iint_{x^2+y^2\le 1} z^2\Big|_{z=f} \cdot \|\vec{N}\|\,dA$$

$$I = \int_0^{2\pi}\int_0^1 (1-r^2)(1-r^2)^{-1/2}\, r\,dr\,d\theta$$

$$I = \int_0^{2\pi}\int_0^1 (1-r^2)^{1/2}\, r\,dr\,d\theta = \frac{2\pi}{3}$$

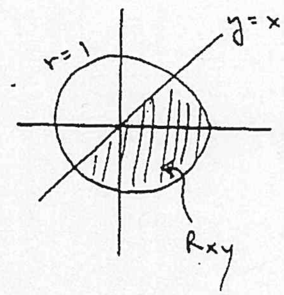
5) $z = f(x,y) = 1$, $\|\vec{N}\| = 1$, $dS = dA$



$$\vec{r}\cdot\vec{r}\Big|_S = x^2+y^2+z^2\Big|_S = x^2+y^2+1$$

$$\iint_S \vec{r}\cdot\vec{r}\,dS = \int_0^1\int_0^1 (x^2+y^2+1)\,dy\,dx = \frac{5}{3}$$

6) $z = 2 - x^2 - y^2$, $\|\vec{N}\| = \sqrt{1+4r^2}$



$$\iint_S dS = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\int_0^1 r\sqrt{1+4r^2}\,dr\,d\theta$$

$$= \frac{\pi}{12}(5\sqrt{5}-1)$$

7) Similar to # 3) , $z = f(x,y) = \sqrt{x^2+y^2}$, $\|\vec{N}\| = \sqrt{2}$

$$\iint_S x^2 z \, dS = \iint_{R_{xy}} \sqrt{2}\, x^2 \sqrt{x^2+y^2}\, dA$$

$$\text{MASS } M = \int_0^{2\pi}\int_1^4 \sqrt{2}\, r^4 \cos^2\theta \, dr\, d\theta = \frac{1023}{5}\pi\sqrt{2}$$

SECTION V

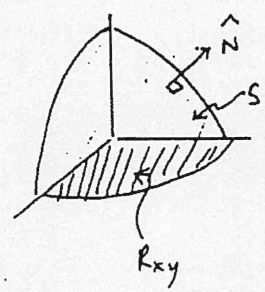
1) $\vec{F} = (z, 0, 0)$, $z = f(x,y) = \sqrt{1-x^2-y^2}$ and

$\vec{N} = (-f_x, -f_y, 1) = \left(\frac{x}{\sqrt{1-r^2}}, \frac{y}{\sqrt{1-r^2}}, 1\right)$ in mixed coord.

$$\iint_S \vec{F}\cdot\hat{N}\, dS = \iint_{R_{xy}} (\sqrt{1-r^2}, 0, 0)\cdot\left(\frac{x}{\sqrt{1-r^2}}, \frac{y}{\sqrt{1-r^2}}, 1\right) dA$$

$$\Phi = \int_0^{2\pi}\int_0^1 r^2 \cos\theta \, dr\, d\theta = 0$$

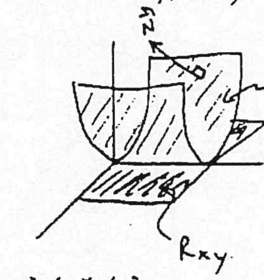
2) $\vec{F} = (0, 0, z)$, $z = f(x,y) = \sqrt{4-x^2-y^2}$, $\vec{N} = \left(\frac{x}{\sqrt{4-r^2}}, \frac{y}{\sqrt{4-r^2}}, 1\right)$



$$\Phi = \iint_S \vec{F}\cdot\hat{N}\, dS = \iint_{R_{xy}} z|_S \, dS$$

$$\Phi = \int_0^{\pi/2}\int_0^2 r\sqrt{4-r^2}\, dr\, d\theta = \frac{4\pi}{3}$$

3) $\vec{F} = (x, y, z)$, $z = f(x,y) = x^2$, $\vec{N} = (-2x, 0, 1)$

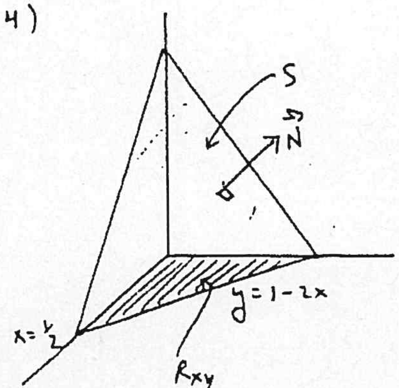


$$\Phi = \iint_{R_{xy}} -2x^2 + 0 + x^2 \, dA$$

$$\Phi = \int_{-2}^{2}\int_0^1 (-x^2)\, dy\, dx = -\frac{16}{3}$$

$-2 < x < 2$
$0 < y < 1$

4)



$z = f(x,y) = 1 - 2x - y$

$\vec{N} = (2, 1, 1)$

$\vec{F}\cdot\vec{N} = (z-x, 0, z+1)\cdot(2,1,1)$

$= 3z - 2x + 1$

$\vec{F}\cdot\vec{N}\big|_S = 4 - 8x - 3y$

$$\Phi = \int_0^{1/2}\int_0^{1-2x} (4 - 8x - 3y)\, dy\, dx = \frac{5}{12}$$

SECTION VI

1) i) $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xy-y & x-x^2 & 0 \end{vmatrix} = (2-3x)\hat{k} \neq \vec{0}$ NOT CONS.

ii) $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y^2-ye^{xy} & 2xy-xe^{xy} & 0 \end{vmatrix} = \vec{0}$ CONSERVATIVE

iii) $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xy & xy & xz \end{vmatrix} = (0,-z,y-x)$ NOT CONS.

2) i) $\phi_x = 3x^2y+y \Rightarrow \phi(x,y) = x^3y+xy+\psi(y)$

$\phi_y = x^3+x-2y = x^3+x+\psi'(y) \Rightarrow \psi'(y) = -2y \Rightarrow \psi = -y^2$

$\phi(x,y) = x^3y+xy-y^2$

ii) $\phi_x = ye^{xy}+\frac{1}{x+2y} \Rightarrow \phi(x,y) = e^{xy}+\log(x+2y)+\psi(y)$

$\phi_y = xe^{xy}+\frac{2}{x+2y} = xe^{xy}+\frac{2}{x+2y}+\psi'(y) \Rightarrow \psi = 0$

$\phi(x,y) = e^{xy}+\log(x+2y)$

iii) $\phi_x = 2xy \Rightarrow \phi(x,y,z) = x^2y + f(y,z)$

$\phi_y = x^2-z = x^2+\frac{\partial f}{\partial y} \Rightarrow f(y,z) = -yz+g(z)$

$\phi_z = 1-\cancel{y} = -\cancel{y}+g'(z) \Rightarrow g(z) = z$

$\phi(x,y,z) = x^2y-yz+z$

3) Like question 2) must compute ϕ so that $\vec{F} = \vec{\nabla}\phi$.

i) $\vec{F}(x,y) = (1+y,1+x) \Rightarrow \phi(x,y) = x+xy+y$

$P = (0,1) \qquad Q = (2,2)$

$\int_P^Q \vec{F}\cdot d\vec{r} = \phi(Q)-\phi(P) = 8-1 = 7$

ii) $\vec{F}(x,y) = \left(\frac{1}{y}, 1-\frac{x}{y^2}\right) \Rightarrow \phi(x,y) = \frac{x}{y}+y$

$P = \vec{r}(0) = (1,2) \qquad Q = \vec{r}(2) = (3,4)$

$\int_P^Q \vec{F}\cdot d\vec{r} = \phi(Q)-\phi(P) = 2\frac{1}{4}$

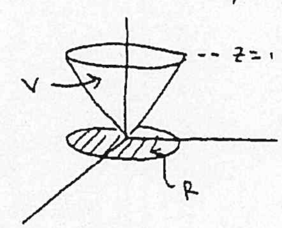
iii) $\vec{F}(x,y,z) = (1,1,2z) \Rightarrow \phi(x,y,z) = x+y+z^2$

$P = (0,0,0) \qquad Q = (1,1,1)$

$\int_P^Q \vec{F}\cdot d\vec{r} = \phi(Q)-\phi(P) = 3$

VII DIVERGENCE THEOREM

1) $\vec{\nabla}\cdot\vec{F} = x^2+y^2+1$



$r < z < 1$

$0 < r < 1$

$0 < \theta < 2\pi$

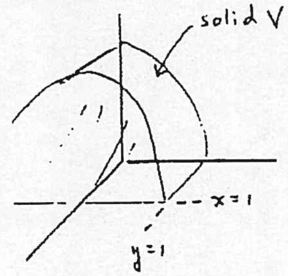
$\Phi = \oiint_S \vec{F}\cdot\hat{N}\,ds$

$\Phi = \iiint_V (x^2+y^2+1)\,dV$

$\Phi = \int_0^{2\pi}\int_0^1\int_r^1 (r^2+1)\,r\,dz\,dr\,d\theta$

$\Phi = \frac{13}{30}\pi$

UESTION 2

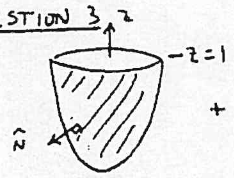


$$\vec{\nabla}\cdot\vec{F} = 3$$

$$\Phi = \oint\!\!\oint_S \vec{F}\cdot\hat{N}\,ds = \iiint_V 3\,dV$$

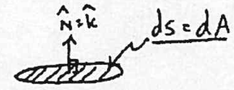
$$\Phi = 3\int_0^1\int_{-1}^1\int_0^{1-y^2} dz\,dy\,dx = 4$$

UESTION 3



S_{BOT}

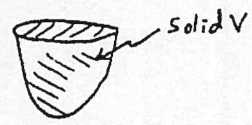
+



S_{TOP}

=

$$\vec{\nabla}\cdot\vec{F} = x^2+y^2 = r^2$$



S

$$\underbrace{\iint_{S_{BOT}} \vec{F}\cdot\hat{N}\,ds}_{\text{want this}} + \iint_{S_{TOP}} \vec{F}\cdot\hat{k}\,dA = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV$$

$$\iint_{S_{BOT}} \vec{F}\cdot\hat{N}\,ds + \int_0^{2\pi}\int_0^1 r^3\sin^2\theta\,dr\,d\theta = \int_0^{2\pi}\int_0^1\int_{r^2}^1 r^3\,dz\,dr\,d\theta$$

$$\iint_{S_{BOT}} \vec{F}\cdot\hat{N}\,ds + \frac{\pi}{4} = \frac{\pi}{6}$$

$$\iint_{S_{BOT}} \vec{F}\cdot\hat{N}\,ds = -\frac{\pi}{12}.$$