Math 333 (2004) Assignment 6

(Due: November 9, 2004 in class)

Maximum 60 points

1. (15) In each of the following an inner product space V with its associated inner product are defined. For the indicated vectors $u, v \in V$ compute $< u, v >$ and $\| u \|$.

a)

$$V = \mathbb{R}^3 \quad , \quad < u, v > \equiv (Au)^T (Av) \quad , \quad A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$
$$u = (1, 1, 1) \quad , \quad v = (2, 0, 1)$$

b)

$$V = C[0, 1] \quad , \quad < u, v > \equiv \int_0^1 u(x)v(x)dx \quad ,$$
$$u = x - 1 \quad , \quad v = x + 1$$

c)

$$V = M_{22} \quad , \quad < u, v > \equiv Tr(u^T v) \quad ,$$
$$u = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad , \quad v = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$$

2. (5) If $V = C[0, 1]$ and $< u, v > \equiv \int_0^1 u(x)v(x)dx$, find all α (if any) for which $u = \alpha - 3x$ and $v = \alpha x + 1$ are orthogonal.

3. (5) Consider the following subspace W of $V = M_{22}$:

$$W = \{u \in M_{22} : u^T = u\}$$

Elements of u are symmetric matrices which always have real eigenvalues. For any $u \in W$ we let λ_1 and λ_2 be the eigenvalues of u. Likewise μ_1 and μ_2 are the eigenvalues of v. Now define

$$< u, v > \equiv \lambda_1 \mu_1 + \lambda_2 \mu_2$$

Is this an inner product on W? Specifically which of the five axioms defining an inner product are satisfied and which (if any) are not?

4. (5) Let $V = P_2$ and define

$$< u, v > = u(0)v(0) + u\left(\frac{1}{2}\right) v\left(\frac{1}{2}\right) + u(1)v(1)$$

Is this an inner product on V? To be clear if $u = ax^2 + bx + c$ then $u(0) = c$, $u(1) = a + b + c$, etc.

5. (10) Let $E = \{e_1, e_2, \dots e_n\}$ be an orthonormal basis for an inner product space V. Prove that if $(u)_E = (u_1, u_2, \dots u_n)$ and $(v)_E = (v_1, v_2, \dots v_n)$ then

$$\| u \| = \sqrt{u_1^2 + u_2^2 + \cdots u_n^2}$$

and

$$< u, v > = u_1 v_1 + \cdots u_n v_n$$

This essentially shows how any inner product can be reduced to the dot product on $\mathbb{R}^n$.

6. (10) For each of the following find a basis for $W^\perp$.

a)

$$W = span\{x^2 - 1, x\} \subset P_2 \quad , \quad < u, v > = \int_0^1 u(x)v(x)dx$$

b)

$$W = span\{(1,2,1,2)^T, (0,1,1,1)^T\} \subset \mathbb{R}^4 \quad , \quad < u, v > = u^T v$$

7. (10) Recall from class the Fredholm alternative Theorem:

Theorem 1 *Let $A \in \mathbb{R}^{n \times n}$. Then, $Ax = b$ has a solution $\Leftrightarrow < v, b > = 0 \ , \ \forall v \in N(A^T)$.*

a) Use this theorem to determine for what $\alpha \in \mathbb{R}$ (if any) the following system has a solution:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= \alpha \\ 3x_1 + x_3 &= 1 - \alpha \\ x_1 - x_2 + x_3 &= 2 + 3\alpha \end{aligned}$$

b) Suppose $b_0 \in col(A)$, $b_1 \in N(A^T)$ and $b_1 \neq 0$ What must $\alpha \in \mathbb{R}$ equal for $Ax = \alpha b_1 + b_0$ to have a solution?