Math 451 (2018) – Final (Take home)

Due: Wed., April 25, 2018. NAME: ______________________

1. [15pts] Derive the natural boundary condition at $x = 1$ for extrema $\bar{y}(x)$ of $J(y)$ over $C^2[1,2]$:

$$J(y) = \frac{1}{2}y(1)^2 + \int_1^2 \left(y'^2 + x^2 y y'\right)\, dx$$

2. [20pts] Find all the eigenvalues λ_n and associated eigenfunctions $\phi_n(x)$ satisfying

$$L\phi_n = \lambda_n \phi_n \qquad , \qquad \phi_n \in D$$

where $Lu = u''$ and $D = \{u \in C^2[0,\pi] : u'(0) = 0, u'(\pi) = 0\}$. You may assume $\lambda \leq 0$.

3. [20pts] Find the solution $u(x)$ of the following integral equation:

$$\int_0^1 k(x,y)u(y)\, dy - u(x) = 20x^3 \qquad , \qquad k(x,y) = 2y + 36xy^2$$

4. [20pts] Find the Green's function $g(x,\zeta)$ for the following (SLP):

$$\begin{aligned} -\frac{d^2u}{dx^2} &= f(x) \qquad , \quad x \in (0,1) \\ u(0) + u'(0) &= 0 \\ u(1) - u'(1) &= 0 \end{aligned}$$

5. [10pts] Let $H(x)$ be the Heaviside step function where $H'(x) = \delta(x)$. Show the following product rule applies in the distributional sense

$$\frac{d}{dx}\left(xH(x)\right) = H(x) + xH'(x)$$

Specifically, show that $T_1 = T_2$ below for all ϕ:

$$T_1 = \left\langle (xH(x))', \phi \right\rangle \qquad , \qquad T_2 = \langle H(x) + xH'(x), \phi \rangle$$

6) a) [15pt] Find the general solution of the following PDE for $u(x,t)$ (integrate in x first):

$$u_{xt} + u_x = x \qquad , \qquad x \in \mathbb{R},\ t > 0$$

Note: $u_{xt} + u_x = \frac{\partial}{\partial x}\left(u_t + u\right)$