Math 451 (2024) – Homework 10

(THROUGHOUT USE INTEGRAL TABLES AND ODE SOLVERS AS NEEDED)

Due: Monday April 15, 2024. NAME: ____________________

1. [5 pts] A bacteria population has density $u(x,t)$ and is both chemotactic and thermotactic. Thus the fluxes are

$$\mathbf{J}_{chemo} = +\alpha u \nabla c$$
$$\mathbf{J}_{thermo} = +\beta u \nabla T$$

where c is the concentration of glucose (food), T is the temperature in the growth medium and α, β are constants. The growth term (source) for the bacteria is $f(u,c)$ for some function f. As a consequence the corresponding loss term for c is $-\lambda f(u,c)$ where λ is yield coefficient. Lastly, c and T have only simple diffusive fluxes.

Write down the (three coupled) partial differential equations for the cases $x \in \mathbb{R}^2$ and $x \in \mathbb{R}$.

2. [5 pts] A random walk with density $u(x,t)$ is defined by

$$u(x,t+\Delta t) = T(\Delta x) = p_{-1}u(x-\Delta x,t) + p_0 u(x,t) + p_1 u(x+\Delta x)$$

where p_k are all the transition probabilities for the walk hence

$$p_{-1} + p_0 + p_1 = 1$$

By Taylor expanding $T(\Delta x)$ in Δx, show that if the walk is biased ($p_{-1} \neq p_1$)

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = D_1 u_x + D_2 u_{xx} + O\left(\frac{\Delta x^3}{\Delta t}\right) \quad , \quad D_1 \neq 0.$$

State what D_1, D_2 are. In the limit $\Delta x, \Delta t \to 0$ with $\Delta x/\Delta t$ fixed and dropping higher order terms we get the advection-diffusion equation for $u(x,t)$:

$$u_t = D_1 u_x + D_2 u_{xx}$$

3. [10 pts] Define the 2-D Laplacian operator

$$Lu = \nabla^2 u \quad , \quad u \in D$$

where D is the set of all functions defined (and sufficiently smooth) on the unit square $R = [0,1]^2$ and satisfying the boundary conditions

$$u_x(0,y) = u_x(1,y) = 0 \qquad , \qquad u(x,0) = u(x,1) = 0$$

These are Neumann boundary conditions at the left and right ($x = 0, 1$) boundaries and Dirichlet at the lower and upper boundaries.

a) Find all the eigenfunctions and eigenvalues of L:

$$L\phi_{mn} = \lambda_{mn}\phi_{mn}$$

where m and n are integers. What is the range for the indices m and n in your typical Fourier expansion

$$f(x,y) = \sum_{m=?}\sum_{n=?} f_{mn}\phi_{mn}(x,y)$$

b) Write out the general (Fourier) solution of

$$\nabla^2 u = f(x,y) \qquad , \qquad u \in D$$

c) (Bonus: 1 point) By rearranging terms and order in your solution above, find the Green's function $g(\mathbf{x}, \mathbf{x}')$ such that

$$u(\mathbf{x}) = \int\int_{\mathbf{R}} \mathbf{g}(\mathbf{x}, \mathbf{x}')\mathbf{f}(\mathbf{x}')\mathbf{dx}'$$

Here $\mathbf{x} = (x,y)$ and $\mathbf{x}' = (x', y')$