Math 451 (2018) – Homework 8

(THROUGHOUT USE INTEGRAL TABLES AND ODE SOLVERS AS NEEDED)

Due: Friday, Feb. 23, 2018. NAME: ____________________

1. [7 pts] Let $\phi_0(x) = 1, \phi_1(x) = \frac{1}{2} - x$. It is easily verified $< \phi_0, \phi_1 >= 0$ in $L^2[0, 1]$.

a) Find constants a, b so that $\phi_2(x) = 6x^2 + ax + b$ is orthogonal to both ϕ_0 and ϕ_1:

$$\begin{aligned} < \phi_0, \phi_2 > &= 0 \\ < \phi_1, \phi_2 > &= 0 \end{aligned}$$

b) Use the orthogonality properties of ϕ_k to find the Fourier coefficients c_k in the expansion

$$f(x) = x^2 + x + 3 = \sum_{k=0}^{k=2} c_k \ \phi_k(x)$$

2. [3 pts] Let $a = y - x, b = z - y, c = z - x$ where x, y, z are elements of any normed inner product space. If $\| \cdot \|$ is the inner product induced norm, i.e., $\| a \|^2 = < a, a >$, prove

$$\| c \|^2 = \| a \|^2 + \| b \|^2$$

is true if $a \perp b$, i.e., $< a, b >= 0$. This is Pythagorus's theorem.

3. [15 pts] Every function $f(x) \in L^2[0, \pi]$ can be written

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n \cos(nx) = \sum_{n \geq 0} c_n \cos nx$$

for the orthogonal set $\{\cos nx\}_{n \geq 0}$ and some Fourier coefficients c_n.

a) Compute c_n for $f(x) = 4 \cos 2x + 7 \cos 7x$.

b) Compute c_n for $f(x) = x$ noting $\cos n\pi = (-1)^n$.

c) Use Parseval's identity and the result in b) to determine a formula for S where

$$S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$$

4. [20 pts] Below are three regular Sturm-Liouville eigenvalue Problems (SLP)

$$\begin{array}{llll} (I) & y'' + \lambda y = 0 & y(0) = 0 & y'(1) = 0 \\ (II) & y'' + \lambda y = 0 & y(0) + y'(0) = 0 & y(1) = 0 \\ (III) & \dfrac{d}{dx}\left(x^2 \dfrac{dy}{dx}\right) + \lambda y = 0 & y(1) = 0 & y(e) = 0 \end{array} \tag{1}$$

Find all eigenvalues λ_n and associated eigenfunctions $y_n(x)$ for each of (I)-(III) above. When possible find an explicit formula for λ_n as in $\lambda_n = n^2$. If you can not find an explicit formula, λ_n will be roots of some function $f(z)$ as in $f(\lambda_n) = 0$. In those cases state what $f(z)$ is.

For each problem consider the cases $\lambda = \mu^2, 0, -\mu^2$

For (II) you may want to write the general solution in the form:

$$y(x) = c_1 \sin(\mu(x-1)) + c_2 \cos(\mu(x-1))$$

Also, (III) is a Cauchy-Euler differential equation in x. Take special note of the $\lambda > \frac{1}{4}$ case.