Method of Characteristics (Intro Example)

The solution of the IVP

(1) $u_t + c u_x = 0 \qquad x \in \mathbb{R},\ t > 0$

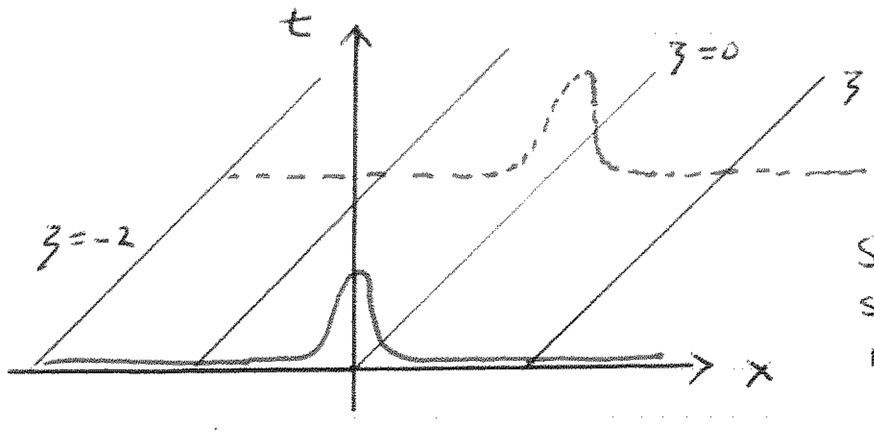
(2) $u(x,0) = f(x)$

is

$$u(x,t) = f(x - ct)$$

The solution $u(x,t)$ is constant on the lines

$$x - ct = \xi \qquad c > 0$$



Shown for some $f(x)$ in red.

These lines/curves are characteristics.

Viewed another way the solution of the PDE problem is

$$u(x,t) = U(s) \qquad s = s(x,t)$$

Suggests a coordinate transformation could be useful.

Linear Case - Method of Characteristics

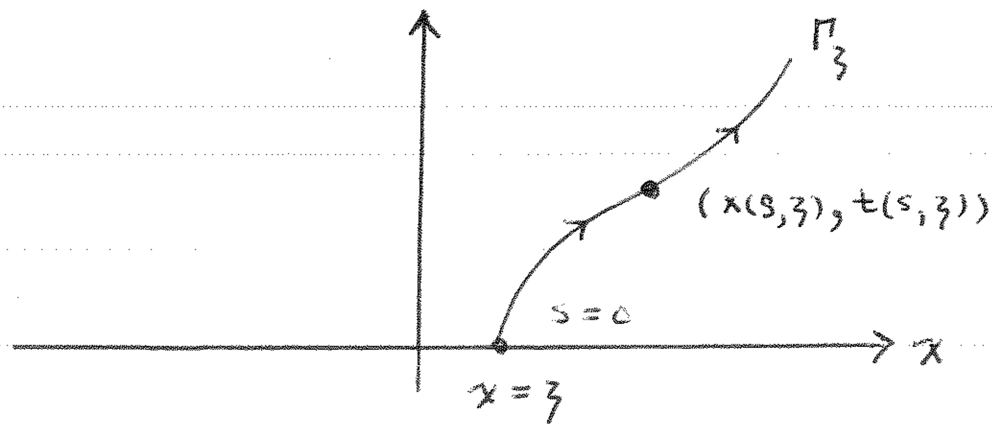
Seek a solution of

(1) $\qquad a u_t + b u_x = c \qquad x \in \mathbb{R}, t > 0$

(2) $\qquad u(x, 0) = f(x)$

where a, b, c can be functions of (x, t)

Characteristics



s parametrizes location on characteristic Γ_ξ

Seek a family of characteristics Γ_ξ such that

(i) $\qquad \Gamma_\xi$ intersects $t=0$ when $s=0$

(ii) $\qquad \dfrac{du}{ds} = c \qquad$ on Γ_ξ

If curves don't intersect they define a coordinate transformation

$$(x, t) \longleftrightarrow (\xi, s)$$

Let $u(x,t)$ be a solution of (1).
Define

$$U(s,\zeta) = u(x(s,\zeta), t(\zeta,s))$$

Then require

$$\underset{\overset{\uparrow}{3}}{\frac{\partial U}{\partial s}} \doteq \underset{\overset{\uparrow}{1}}{\frac{\partial t}{\partial s}} u_t + \underset{\overset{\uparrow}{2}}{\frac{\partial x}{\partial s}} u_x = \underset{\overset{\uparrow}{1}}{a} u_t + \underset{\overset{\uparrow}{2}}{b} u_x = \underset{\overset{\uparrow}{3}}{c}$$

Therefore along characteristics

$$(3) \qquad \frac{dt}{ds} = a(x,t)$$

$$(4) \qquad \frac{dx}{ds} = b(x,t)$$

$$(5) \qquad \frac{du}{ds} = c(x,t)$$

Characteristic equations

The requirement that Γ_ζ intersect $t=0$ at $s=0$ and that u satisfy the initial conditions imply

$$(6) \qquad t(0,\zeta) = 0$$

$$(7) \qquad x(0,\zeta) = 0$$

$$(8) \qquad u(0,\zeta) = f(\zeta)$$

Solve ODE system (3)-(5) subject to (6)-(8) and re-express u in terms of (x,t).

EXAMPLE

(1) $$u_t + x u_x = t$$

(2) $$u(x,0) = f(x)$$

Characteristic Eqns

(3) $$\frac{dt}{ds} = 1 \qquad \frac{dx}{ds} = x \qquad \frac{du}{ds} = t$$

General soln of first two

$$t = s + c_1 \qquad x = c_2 e^s$$

The requirement that $t(0,\xi) = 0$, $x(0,\xi) = \xi \Rightarrow$

(4) $$t = s \qquad x = \xi e^s$$

Now see an explicit inverse

(5) $$s = t \qquad \xi = x e^{-t}$$ } can use to sketch each Γ_ξ for $\xi \in \mathbb{R}$.

Given (4) the ODE for u is

$$\frac{du}{ds} = t = s$$

whose general soln is

$$u = \tfrac{1}{2} s^2 + c_3$$

Using $u(0,\xi) = f(\xi)$ so

$$u = \tfrac{1}{2} s^2 + f(\xi)$$

or,

$$u(x,t) = \tfrac{1}{2} t^2 + f(x e^{-t})$$

EXAMPLE

$$(1) \qquad x u_t - t u_x = xt$$

$$(2) \qquad u(x,0) = f(x)$$

Solution of

$$\frac{dt}{ds} = x \qquad t(0,\zeta) = 0$$

$$\frac{dx}{ds} = -t \qquad x(0,\zeta) = \zeta$$

is

$$(3) \qquad t = \zeta \sin(s) \qquad x = \zeta \cos(s)$$

which can be inverted (for most x, t)

$$\zeta = \pm\sqrt{x^2+t^2} \qquad s = \arctan\left(\frac{t}{x}\right)$$

Then

$$\frac{du}{ds} = xt = \zeta^2 \sin(s)\cos(s) \qquad u(0,\zeta) = f(\zeta)$$

yields (for $\zeta > 0$)

$$u = \tfrac{1}{2}\zeta^2 \sin^2(s) + f(\zeta)$$

$$u = \tfrac{1}{2} t^2 + f\left(\sqrt{x^2+t^2}\right)$$

Characteristics are circles from (3)