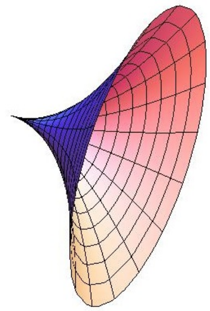
What is the surface area of the figure obtained by rotating $y = x^2$ about the x-axis on the interval $[0,1]$?



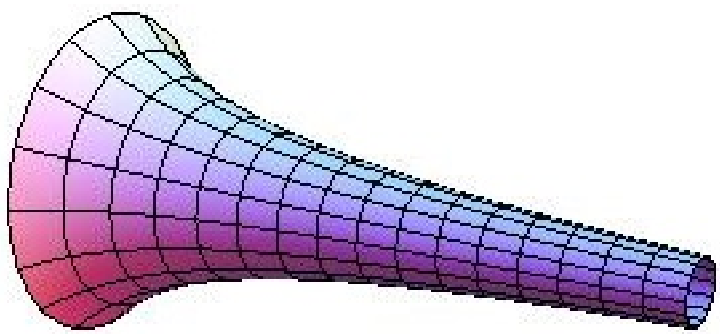
$$\begin{aligned}
S &= \int_a^b 2\pi * (radius) * (arclength) \\
&= 2\pi \int_0^1 x^2\sqrt{1+4x^2}\ dx \\
&= 2\pi \int_0^{\arctan 2} \frac{1}{4} tan^2 u * \sec u * \underbrace{\frac{1}{2}\sec^2 u\ du}_{dx} && 2x = \tan u \\
&= \frac{\pi}{4}\int_0^{\arctan 2} \left(\sec^2 u - 1\right)\sec^3 u\ du \\
&= \frac{\pi}{4}\int_0^{\arctan 2} \left(\sec^5 u - \sec^3 u\right)\ du \\
&= \frac{\pi}{4}\left[\frac{1}{4}\tan u \sec^3 u + \frac{3}{4}\int \sec^3 u\ du - \int \sec^3 u\ du\right]_0^{\arctan 2} \\
&= \frac{\pi}{4}\left[\frac{1}{4}\tan u \sec^3 u - \frac{1}{4}\int \sec^3 u\ du\right]_0^{\arctan 2} \\
&= \frac{\pi}{4}\left[\frac{1}{4}\tan u \sec^3 u - \frac{1}{4}\left(\frac{1}{2}\sec u \tan u + \frac{1}{2}\ln\left|\sec u + \tan u\right|\right)\right]_0^{\arctan 2} \\
&= \frac{\pi}{4}\left[\frac{1}{2}x(4x^2+1)^{3/2} - \frac{1}{4}x\sqrt{4x^2+1} - \frac{1}{8}\ln\left|\sqrt{4x^2+1}+2x\right|\right]_0^1 && u = \arctan 2x \\
&= \frac{\pi}{4}\left[\begin{array}{l}\left(\frac{1}{2}*1*5^{3/2} - \frac{1}{4}*1*\sqrt{5} - \frac{1}{8}\ln\left|\sqrt{5}+2\right|\right) \\ -\left(\frac{1}{2}*0*(0+1)^{3/2} - \frac{1}{4}*0*\sqrt{0+1} - \frac{1}{8}\ln\left|\sqrt{0+1}+2*0\right|\right)\end{array}\right] \\
&= \frac{\pi}{4}\left[\left(\frac{5^{3/2}}{2} - \frac{\sqrt{5}}{4} - \frac{1}{8}\ln\left|\sqrt{5}+2\right|\right) - \left(0 - 0 - \frac{1}{8}\ln|1|\right)\right] \\
&= \frac{\pi}{4}\left[\frac{5^{3/2}}{2} - \frac{\sqrt{5}}{4} - \frac{1}{8}\ln\left|\sqrt{5}+2\right|\right] \\
&\approx 3.80973\ldots
\end{aligned}$$

Noting the identity $\sinh^{-1} x = \ln\left|\sqrt{1+x^2}+x\right|$, we could also express our answer as

$$\frac{\pi}{4}\left[\frac{5^{3/2}}{2} - \frac{\sqrt{5}}{4} - \frac{1}{8}\sinh^{-1}(2)\right]$$

“An ice cream cone where you can eat the ice cream but not the cone.”

Also known as Gabriel’s Horn or Torricelli’s Trumpet, consider the surface obtained by rotating $y = 1/x$ on $[1, \infty)$ about the x-axis.



The volume (i.e., amount of ice cream) is given by:

$$\begin{aligned} V &= \int_a^b \pi(radius)^2 \\ &= \pi \int_1^\infty \frac{1}{x^2}\,dx \\ &= \pi \lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx \\ &= \pi \lim_{R\to\infty} \left.\frac{-1}{x}\right|_1^R \\ &= \pi \lim_{R\to\infty} \left[\frac{-1}{R} - \frac{-1}{1}\right] \\ &= \pi * 1 \\ &= \pi \end{aligned}$$

So we have a finite volume, and thus our “cone” holds a finite ammount of ice cream. Now for surface area (i.e., amount of waffle cone):

$$\begin{aligned} S &= \int_a^b 2\pi(radius) * (arc\ length) \\ &= \int_1^\infty 2\pi\left(\frac{1}{x}\right)\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\,dx \\ &= \lim_{R\to\infty} 2\pi \int_1^R \left(\frac{1}{x}\right)\sqrt{1+\frac{1}{x^4}}\,dx \end{aligned}$$

Let’s tackle this by finding an indefinite integral first and then dealing with bounds:

$$2\pi\int\left(\frac{1}{x}\right)\sqrt{1+\frac{1}{x^4}}\,dx = 2\pi\int\left(\frac{1*x^4}{x*x^4}\right)\sqrt{1+\frac{1}{x^4}}\,dx$$

$$
\begin{aligned}
&= 2\pi \int x^4 \sqrt{1+\frac{1}{x^4}} * \left(\frac{1}{x^5}\right)\, dx \\
&= \frac{-\pi}{2} \int \left(\frac{1}{u}\right) \sqrt{1+u} * \left(\frac{-1}{4}\right)\, du && u = 1/x^4 \\
&= \frac{-\pi}{2} \int \frac{\sqrt{1+u}}{u}\, du \\
&= \frac{-\pi}{2} \int \frac{\sqrt{1+u}}{u} * \frac{\sqrt{1+u}}{\sqrt{1+u}}\, du \\
&= -\pi \int \frac{s^2}{s^2-1}\, ds && s = \sqrt{1+u} \\
&= -\pi \int \left(1 + \frac{1/2}{s-1} - \frac{1/2}{s+1}\right)\, ds && \text{division, part. frac.} \\
&= -\pi \left[s + \frac{1}{2}\ln|s-1| - \frac{1}{2}\ln|s+1|\right] + C \\
&= -\pi \left[\sqrt{1+u} + \frac{1}{2}\ln\left|\sqrt{1+u}-1\right| - \frac{1}{2}\ln\left|\sqrt{1+u}+1\right|\right] + C \\
&= -\pi \left[\sqrt{1+\frac{1}{x^4}} + \frac{1}{2}\ln\left|\sqrt{1+\frac{1}{x^4}}-1\right| - \frac{1}{2}\ln\left|\sqrt{1+\frac{1}{x^4}}+1\right|\right] + C \\
&= \frac{-\pi}{2} \left[2\sqrt{1+\frac{1}{x^4}} + \ln\left|\sqrt{1+\frac{1}{x^4}}-1\right| - \ln\left|\sqrt{1+\frac{1}{x^4}}+1\right|\right] + C \\
&= \frac{-\pi}{2} \left[\frac{2\sqrt{1+x^4}}{x^2} + \ln\left|\frac{\sqrt{1+x^4}}{x^2}-1\right| - \ln\left|\frac{\sqrt{1+x^4}}{x^2}+1\right|\right] + C \\
&= \frac{-\pi}{2} \left[\frac{2\sqrt{1+x^4}}{x^2} + \ln\left|\frac{\sqrt{1+x^4}-x^2}{x^2}\right| - \ln\left|\frac{\sqrt{1+x^4}+x^2}{x^2}\right|\right] + C \\
&= \frac{-\pi}{2} \left[\frac{2\sqrt{1+x^4}}{x^2} + \ln\left|\frac{\sqrt{1+x^4}-x^2}{\sqrt{1+x^4}+x^2}\right|\right] + C \\
&= \frac{-\pi\sqrt{1+x^4}}{x^2} + \frac{\pi}{2}\ln\left|\frac{\sqrt{1+x^4}+x^2}{\sqrt{1+x^4}-x^2}\right| + C
\end{aligned}
$$

Now that we have an indefinite integral, let's evalute our original definite integral:

$$
\begin{aligned}
S &= \int_1^\infty 2\pi \left(\frac{1}{x}\right) \sqrt{1+\left(\frac{-1}{x^2}\right)^2}\, dx \\
&= \lim_{R\to\infty} 2\pi \int_1^R \left(\frac{1}{x}\right) \sqrt{1+\frac{1}{x^4}}\, dx \\
&= \lim_{R\to\infty} \left[\frac{-\pi\sqrt{1+x^4}}{x^2} + \frac{\pi}{2}\ln\left|\frac{\sqrt{1+x^4}+x^2}{\sqrt{1+x^4}-x^2}\right|\right]_1^R
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{R\to\infty}\left[\left(\frac{-\pi\sqrt{1+R^4}}{R^2}+\frac{\pi}{2}\ln\left|\frac{\sqrt{1+R^4}+R^2}{\sqrt{1+R^4}-R^2}\right|\right)-\left(\frac{-\pi\sqrt{1+1^4}}{1^2}+\frac{\pi}{2}\ln\left|\frac{\sqrt{1+1^4}+1^2}{\sqrt{1+1^4}-1^2}\right|\right)\right]\\
&= \lim_{R\to\infty}\left[\frac{-\pi\sqrt{1+R^4}}{R^2}+\frac{\pi}{2}\ln\left|\frac{\sqrt{1+R^4}+R^2}{\sqrt{1+R^4}-R^2}\right|+\pi\sqrt{2}-\frac{\pi}{2}\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|\right]\\
&= \lim_{R\to\infty}\left[\frac{-\pi R^2\sqrt{\frac{1}{R^4}+1}}{R^2}+\frac{\pi}{2}\ln\left|\frac{R^2\left(\sqrt{\frac{1}{R^4}+1}+1\right)}{R^2\left(\sqrt{\frac{1}{R^4}+1}-1\right)}\right|+\pi\sqrt{2}-\frac{\pi}{2}\ln(3+2\sqrt{2})\right]\\
&= \lim_{R\to\infty}\left[-\pi\sqrt{\frac{1}{R^4}+1}+\frac{\pi}{2}\ln\left|\frac{\left(\sqrt{\frac{1}{R^4}+1}+1\right)}{\left(\sqrt{\frac{1}{R^4}+1}-1\right)}\right|+\pi\sqrt{2}-\frac{\pi}{2}\ln(3+2\sqrt{2})\right]\\
&= -\pi+\frac{\pi}{2}\infty+\pi\sqrt{2}-\frac{\pi}{2}\ln(3+2\sqrt{2})\\
&= \infty
\end{aligned}
$$

and thus we have an infinite surface area.

Are you adequately convinced that you *really* don't want to do this on a test? Surely there's an easier way...let's go back to our integral of interest:

$$S=\int_1^\infty 2\pi\left(\frac{1}{x}\right)\sqrt{1+\frac{1}{x^4}}\,dx$$

For $1\leq x<\infty$,

$$
\begin{aligned}
1 \leq&\sqrt{1+\frac{1}{x^4}}\leq\sqrt{2}\leq 2\\
\implies 2\pi \leq& 2\pi\sqrt{1+\frac{1}{x^4}}\leq 2*2\pi\\
\implies \frac{2\pi}{x} \leq& \frac{2\pi}{x}\sqrt{1+\frac{1}{x^4}}\leq\frac{4\pi}{x}
\end{aligned}
$$

Therefore,

$$\int_1^\infty\frac{2\pi}{x}\sqrt{1+\frac{1}{x^4}}\,dx\geq\int_1^\infty\frac{2\pi}{x}\,dx=2\pi\int_1^\infty\frac{1}{x}\,dx.$$

Since we know that $\int_1^\infty \frac{1}{x}\,dx$ diverges to $+\infty$, we have that $\int_1^\infty\frac{2\pi}{x}\sqrt{1+\frac{1}{x^4}}\,dx$ diverges to $+\infty$ by the comparison theorem. Therefore, we have an infinite surface area.