

Improper Integrals, Arclength, Surface Area, Centroids

I. Decide if the following are convergent or divergent integrals. Evaluate them if they are convergent.

$$\int_2^5 \frac{dx}{\sqrt{3x-6}} = 6 \text{ (convergent)} \quad (1)$$

$$\int_0^\infty \frac{dx}{x^2+4} = \frac{\pi}{4} \text{ (convergent)} \quad (2)$$

$$\int_0^1 \ln(x) dx = -1 \text{ (convergent)} \quad (3)$$

$$\int_0^\infty \frac{e^x}{e^{2x}+1} dx = \frac{\pi}{4} \text{ (convergent, } u = e^x) \quad (4)$$

$$\int_{-\infty}^\infty \frac{x}{x^4+1} dx = 0 \text{ (convergent)} \quad (5)$$

$$\int_0^\infty xe^x dx \quad \text{(divergent)} \quad (6)$$

II. Use the Comparison Test to decide if the following converge or diverge:

$$\int_1^\infty \frac{e^x}{\sqrt{x}} dx \quad \text{diverges} \quad \frac{e^x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}} \text{ for } x > 1 \quad (7)$$

$$\int_2^\infty \frac{\sqrt{x}}{(x^2+1)} dx \quad \text{converges} \quad 0 \leq \frac{\sqrt{x}}{(x^2+1)} \leq \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}} \text{ for } x > 2 \quad (8)$$

$$\int_0^\infty \frac{\cos^2(x)}{x^2+2} dx \quad \text{converges} \quad 0 \leq \frac{\cos^2(x)}{x^2+2} \leq \frac{1}{x^2+2} \leq \frac{1}{x^2} \text{ for } x > 0 \quad (9)$$

$$\int_0^\infty \frac{\arctan(x)}{x^3+2} dx \quad \text{converges} \quad 0 \leq \frac{\arctan(x)}{x^3+2} \leq \frac{\pi}{2} \frac{1}{x^3+2} \leq \frac{\pi}{2} \frac{1}{x^3} \text{ for } x > 0 \quad (10)$$

$$\int_1^\infty \frac{\sin^2(x)+1}{\sqrt{x}+1} dx \quad \text{diverges} \quad \frac{\sin^2(x)+1}{\sqrt{x}+1} \geq \frac{1}{\sqrt{x}+1} \geq \frac{1}{2\sqrt{x}} \text{ for } x \geq 1 \quad (11)$$

III. Find the arclength of the following curves:

$$y = \frac{1}{3}(1+2x)^{(3/2)} \quad 0 < x < 1 \quad \text{ANS: } L = \int_0^1 \sqrt{2+2x} dx = \frac{8}{3} - \frac{2\sqrt{2}}{3} \quad (12)$$

$$y = \frac{1}{3}(x^2+2)^{(3/2)} \quad 0 < x < 1 \quad \text{ANS: } L = \int_0^1 (1+x^2) dx = \frac{4}{3} \quad (13)$$

$$y = \frac{3}{\sqrt{2}}x^{(2/3)} \quad 1 < x < 8 \quad \text{ANS: } L = \int_1^8 \frac{\sqrt{x^{2/3}+2}}{x^{1/3}} dx = 3^{3/2}(2^{3/2}-1) \quad u = x^{2/3} \quad (14)$$

$$y = \frac{x^6+8}{16x^2} \quad 1 < x < 2 \quad \text{ANS: } L = \int_1^2 \frac{4+x^6}{4x^3} dx = \frac{21}{16} \quad (15)$$

$$y = \log(\sec(x)) \quad 0 < x < \pi/6 \quad \text{ANS: } L = \int_0^{\pi/6} \sec(x) dx = \frac{1}{2}\ln(3) \quad (16)$$

$$y = \int_0^x \sqrt{e^t-1} dt \quad 0 < x < \ln(4) \quad \text{ANS: } L = \int_0^{\ln(4)} 2e^{x/2} dx = 2 \quad (17)$$

IV. Find the surface area of the surface formed by revolving the following curves about the x -axis:

$$y = \sqrt{3}x \quad 1 < x < 3 \quad \text{ANS :} \quad S = \int_1^3 4\pi\sqrt{3} x \, dx = 16\pi\sqrt{3} \quad (18)$$

$$y = \sqrt{3x+1} \quad -\frac{1}{3} < x < 1 \quad \text{ANS :} \quad S = \int_{-1/3}^1 \pi\sqrt{12x+13} \, dx = \frac{49}{9}\pi \quad (19)$$

$$y = \frac{2}{3}x^3 \quad 0 < x < 1 \quad \text{ANS :} \quad S = \int_0^1 \frac{4}{3}\pi x^3 \sqrt{1+4x^2} \, dx = \frac{1}{18} (5^{3/2} - 1) \pi \quad (20)$$

$$y = \sqrt{25-x^2} \quad 2 < x < 3 \quad \text{ANS :} \quad S = \int_2^3 10\pi \, dx = 10\pi \quad (21)$$

$$y = \cos(x) \quad 0 < x < \frac{\pi}{4} \quad \text{ANS :} \quad S = \int_0^{\pi/4} 2\pi\cos(x)\sqrt{1+\sin^2(x)} \, dx \quad (22)$$

$$y = \frac{3}{2}x^{2/3} \quad 0 < x < 1 \quad \text{ANS :} \quad S = \int_0^1 3\pi x^{1/3} \sqrt{1+x^{2/3}} \, dx = \frac{6}{5} (\sqrt{2}+1) \pi \quad (23)$$

$$\text{use } u = x^{1/3} \quad (24)$$

$$y = \frac{1}{2}x^2 \quad 0 < x < 1 \quad \text{ANS :} \quad S = \int_0^1 \pi x^2 \sqrt{1+x^2} \, dx = \frac{\pi}{8} (\ln(\sqrt{2}-1) + 3\sqrt{2}) \quad (25)$$

8. [10pts] Find the x and y coordinates of the center of mass of the centroid (region) bounded by the curves:

$$y = \frac{1}{2}x^2, y = 0, x = 0, x = 2, \quad \bar{x} = \frac{3}{2}, \bar{y} = \frac{3}{5}$$

$$y = e^x, y = 0, x = 0, x = \ln(2) \quad \bar{x} = 2 \ln(2) - 1, \bar{y} = \frac{3}{4}$$

$$y = \sin(x), y = 0, x = 0, x = \frac{\pi}{2} \quad 0 < x < \frac{\pi}{2} \quad \bar{x} = 1, \bar{y} = \frac{\pi}{8}$$

Indefinite integrals and Trigonometric Identities

Some commonly used trigonometric identities are:

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ \tan^2(x) + 1 &= \sec^2(x) \\ \cos^2(x) &= \frac{1}{2}(1 + \cos(2x)) \\ \sin^2(x) &= \frac{1}{2}(1 - \cos(2x)) \\ \sin(2x) &= 2 \sin(x) \cos(x) \\ \sin(x) \cos(y) &= \frac{1}{2}(\sin(x+y) + \sin(x-y)) \\ \cos(x) \cos(y) &= \frac{1}{2}(\cos(x+y) + \cos(x-y)) \\ \sin(x) \sin(y) &= \frac{1}{2}(\cos(x-y) - \cos(x+y)) \end{aligned}$$

Some commonly integrals worth noting include:

$$\begin{aligned} \int \frac{1}{u^2 + 1} du &= \arctan(u) + c \\ \int \frac{1}{\sqrt{1 - u^2}} du &= \arcsin(u) + c \\ \int \tan(u) du &= \ln|\sec(u)| + c \\ \int \sec(u) du &= \ln|\sec(u) + \tan(u)| + c \end{aligned}$$