Math 172 - Sequence and Series Review Problems - Sketch of Solutions

Notes regarding the sketches below:

- The solutions given below are only sketches of solutions, they are not written in a complete manor. Rather, they are indicative of one possible method of solution.
- All inequalites are valid for sufficiently large n.
- In the problems and solutions, we use $\sum a_n$ instead of $\sum_{n=1}^{\infty} a_n$, or $\sum_{n=k}^{\infty} a_n$ for any k, if the distinction is immaterial.

1. Determine if the following series converge or diverge. Determine the sum for convergent series.

   (a) $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^{2k}} = \frac{-4}{5}$ (Geometric Series: $|r| = 1/4 < 1$)

   (b) $\sum_{k=0}^{\infty} 2^k 3^{2-k} = 27$ (Geometric Series: $|r| = 2/3 < 1$)

   (c) $\sum_{k=1}^{\infty} \frac{3^{2k+1}}{2^{3k+1}}$ diverges (Geometric Series: $|r| = 9/8 \geq 1$)

   (d) $\sum_{k=1}^{\infty} \cos k$ diverges by the Test for Divergence.

   (e) $\sum_{k=2}^{\infty} \frac{2}{k^2-1} = \frac{3}{2}$ (Telescoping Series: $\frac{2}{k^2-1} = \frac{1}{k-1} - \frac{1}{k+1}$)

   (f) $\sum_{k=2}^{\infty} \frac{6}{4k^2-9} = \frac{23}{15}$ (Telescoping Series: $\frac{6}{4k^2-9} = \frac{1}{2k-3} - \frac{1}{2k+3}$)

2. Use the integral test to determine whether the following converge. Make sure you state/verify the hypotheses.

   (a) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges (Could also compare $0 \leq \frac{\ln n}{n^2} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ and $\sum \frac{1}{n^{3/2}}$ converges.)

   (b) $\sum_{n=2}^{\infty} \frac{n^2}{e^{n^3}}$ converges (Could also compare $0 \leq \frac{n^2}{e^{n^3}} \leq \frac{n^2}{e^{3n}} = \frac{n}{e^n}\frac{n}{e^n}\frac{1}{e^n} \leq \frac{1}{e^n}$ and $\sum \frac{1}{e^n}$ converges. Could also use the Ratio Test, $\rho = 0$, or the Root Test $L = 0$.)

   (c) $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$ diverges (This series requires the Integral Test.) [or the Cauchy Condensation Test :]

3. Determine if the following series converge or diverge.

   (a) $\sum \frac{n+2}{n^4}$ converges, Limit Comparison Test (L.C.T) with $\sum \frac{1}{n^3}$ which converges, or directly compare $0 \leq \frac{n+2}{n^4} \leq \frac{2n}{n^4} = \frac{2}{n^3}$ and $2\sum \frac{1}{n^3}$ converges.

   (b) $\sum \frac{\cos(\pi n) \ln n}{n}$ converges, Alternating Series Test (A.S.T.) [note: $\cos(\pi n) = (-1)^n$]

   (c) $\sum \frac{\sqrt[4]{n^3-1}}{n^2+n}$ converges, direct comparison $0 \leq \frac{\sqrt[4]{n^3-1}}{n^2+n} \leq \frac{n^{3/4}}{n^2} = \frac{1}{n^{5/4}}$ and $\sum \frac{1}{n^{5/4}}$ converges

   (d) $\sum \frac{2^n}{3^n+4^n}$ converges, direct comparison $0 \leq \frac{2^n}{3^n+4^n} \leq \frac{2^n}{3^n}$ and $\sum (\frac{2}{3})^n$ converges

(e) $\sum \dfrac{2n+1}{3n^2+4}$ diverges, L.C.T. with $\sum \frac{1}{n}$ which diverges, or directly compare $\frac{2n+1}{3n^2+4} \geq \frac{2n}{3n^2+n^2} = \frac{1}{2n}$ and $\frac{1}{2}\sum \frac{1}{n}$ diverges.

(f) $\sum \left(1 + \dfrac{(-1)^n}{n}\right)$ diverges, Test for Divergence

4. Determine if the following series absolutely converge, conditionally converge, or diverge.

(a) $\sum \dfrac{\sin n}{n^2+1}$ converges absolutely, direct comparison $0 \leq |\frac{\sin n}{n^2+1}| \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges

(b) $\sum \dfrac{(2n)!}{n^5 5^n n!}$ diverges, Ratio Test $\rho = \infty$

(c) $\sum \dfrac{n^2 2^n}{(2n+1)!}$ converges absolutely, Ratio Test $\rho = 0$

(d) $\sum \dfrac{(-1)^n}{n+\sqrt{n}}$ conveges conditionally, A.S.T. shows convergence, does not absolutely converge by comparison $\frac{1}{n+\sqrt{n}} \geq \frac{1}{n+n} = \frac{1}{2n} \geq 0$ and $\sum \frac{1}{2n} = \frac{1}{2}\sum \frac{1}{n}$ diverges

(e) $\sum \left(\dfrac{2n+1}{3n-2}\right)^{2n}$ converges absolutely, Root Test $L = \frac{4}{9}$

(f) $\sum \dfrac{(-1)^n n^3}{(n^2+3)^2}$ converges conditionally, A.S.T shows convergence, does not absolutely converge by L.C.T with $\sum \frac{1}{n}$

5. Find the radius of convergence and interval of convergence for the following.

(a) $\sum \dfrac{(x-1)^n}{n5^n}$ $R = 5$, $I = [-4, 6)$

(b) $\sum \dfrac{(2x)^n}{n^n}$ $R = \infty$, $I = (-\infty, \infty)$

(c) $\sum (5x)^n$ $R = 1/5$, $I = (-1/5, 1/5)$

(d) $\sum n(2-x)^n$ $R = 1$, $I = (1, 3)$

(e) $\sum \dfrac{(x+4)^n}{n^4}$ $R = 1$, $[-5, -3]$

(f) $\sum \dfrac{x^n}{n!}$ $R = \infty$, $I = (-\infty, \infty)$

6. True/False Questions. If false, provide a counterexample.

(a) T/F If $\lim_{n\to\infty} |a_n| = 0$ then $\{a_n\}$ converges. True. $|a_n| \to 0 \implies a_n \to 0$, Squeeze Theorem.

(b) T/F If $\lim_{n\to\infty} |a_n| = 0$ then $\sum a_n$ converges. False. $1/n \to 0$ but $\sum 1/n$ diverges.

(c) T/F If $\lim_{n\to\infty} a_n = 1$ then $\{a_n\}$ converges. True, definition.

(d) T/F If $\lim_{n\to\infty} |a_n| = 1$ then $\{a_n\}$ converges. False. Take $a_n = (-1)^n$.

(e) T/F If $\lim_{n\to\infty} |a_n| = 1$ then $\sum a_n$ converges. False. Test for Divergence.

(f) T/F If $\lim_{n\to\infty} a_n = \infty$ then $\{a_n\}$ converges. False. The sequence **diverges** to infinity.

(g) T/F The power series $\sum a_n x^n$ can diverge for all values of x. False. Power series always converge at their center, in this case $x = 0$.

(h) T/F The power series $\sum a_n x^n$ can converge for all values of x. True.

(i) T/F If the power series $\sum a_n x^n$ converges for $x = -2$ then the series converges for $x = 1$. True. Convergence at $x = -2$ implies the radius of convergence is at least 2.

(j) T/F If $\sum a_n$ converges conditionally then $\sum |a_n|$ converges. False. By definition, a conditionally convergent series is not absolutely convergent.