Conditionally Convergent: What's the big deal?

- Riemann proved that:

> If $\sum a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\sum a_n$ that has a sum equal to r.

- That's VERY WEIRD! Let's collect some data using the alternating harmonic.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15}\cdots$

Overestimates:	Underestimates:
$s_1 = 1$	$s_2 = 1 - \frac{1}{2} = .5$
$s_3 = 1 - \frac{1}{2} + \frac{1}{3} \approx .833333$	$s_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \approx .583333$
$s_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \approx .783333$	$s_6 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \approx .616667$
$s_7 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \approx .759524$	$s_8 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \approx .634524$
$s_9 \approx .745635$	$s_{10} \approx .645635$
$s_{19} \approx .718771$	$s_{20} \approx .668771$
$s_{29} \approx .710091$	$s_{30} \approx .676758$
$s_{49} \approx .703247$	$s_{50} \approx .683247$
$s_{99} \approx .698172$	$s_{100} \approx .688172$
$s_{199} \approx .695653$	$s_{200} \approx .690653$
$s_{499} \approx .694148$	$s_{500} \approx .692148$
$s_{999} \approx .693647$	$s_{1000} \approx .692647$

Note: The sum appears to be $s \approx 0.693147$. It is actually $s = \ln 2$.

2. Change order: $= 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \frac{1}{6} + \frac{1}{9} + \frac{1}{11} - \frac{1}{8} + \frac{1}{13} - \frac{1}{10} + \frac{1}{15} + \frac{1}{17} - \frac{1}{12}\cdots$

$s_1 = 1$	$s_{250} \approx .893385$
$s_2 = 1 - \frac{1}{2} = .5$	$s_{500} \approx .894631$
$s_5 = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} \approx .783333$	$s_{1000} \approx .895255$
$s_{10} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \frac{1}{6} + \frac{1}{9} + \frac{1}{11} - \frac{1}{8} \approx .836544$	$s_{2500} \approx .895630$
$s_{20} \approx .865424$	$s_{5000} \approx .895755$
$s_{30} \approx .875400$	$s_{7500} \approx .895796$
$s_{50} \approx .883507$	$s_{10,000} \approx .895817$
$s_{100} \approx .889662$	$s_{25,000} \approx .895855$

Note: This is definitely a different sum. This series was created to have a sum of 0.9.

- WOW!!

Absolutely Convergent: Is this better?

- Let's collect some data using an absolutely convergent, alternating p-series with $p = 2$.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} - \frac{1}{10^2} + \frac{1}{11^2} - \frac{1}{12^2} + \frac{1}{13^2}\cdots$

Overestimates:	Underestimates:
$s_1 = 1$	$s_2 = 1 - \frac{1}{2^2} = .75$
$s_3 = 1 - \frac{1}{2^2} + \frac{1}{3^2} \approx .861111$	$s_4 = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \approx .798611$
$s_5 = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} \approx .838611$	$s_6 = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} \approx .810833$
$s_7 = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} \approx .831241$	$s_8 = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} \approx .815616$
$s_9 \approx .827962$	$s_{10} \approx .817962$
$s_{19} \approx .823779$	$s_{20} \approx .821279$
$s_{29} \approx .823041$	$s_{30} \approx .821930$
$s_{49} \approx .822671$	$s_{50} \approx .822271$
$s_{99} \approx .822518$	$s_{100} \approx .822418$
$s_{499} \approx .822469$	$s_{500} \approx .822465$
$s_{999} \approx .822468$	$s_{1000} \approx .822467$

Note: This sum appears to be $s \approx 0.8224675$. It is actually $s = \frac{\pi^2}{12}$.

2. Change order: $= 1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{4^2} + \frac{1}{7^2} - \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{11^2} - \frac{1}{8^2} + \frac{1}{13^2} - \frac{1}{10^2}\cdots$

$s_1 = 1$	$s_{250} \approx .823288$
$s_2 = 1 - \frac{1}{2^2} = .75$	$s_{500} \approx .822881$
$s_5 = 1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{4^2} \approx .838611$	$s_{1,000} \approx .822674$
$s_{10} = 1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{4^2} + \frac{1}{7^2} - \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{11^2} - \frac{1}{8^2} \approx .836227$	$s_{2,500} \approx .822550$
$s_{20} \approx .831024$	$s_{5,000} \approx .822509$
$s_{30} \approx .828571$	$s_{7,500} \approx .822495$
$s_{50} \approx .826327$	$s_{10,000} \approx .822488$
$s_{100} \approx .824473$	$s_{25,000} \approx .822475$

Note: Though it takes a long time, it appears this sum is the same.

- **For finite sums, addition is commutative! But, only <u>absolutely convergent</u> series (infinite sums) obey that rule!!**