Summary of Series Convergence/Divergence Theorems

A partial sum s_n is defined as the sum of terms of a sequence $\{a_k\}$. For instance,

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

If the partial sums s_n converge then we say the infinite series converges and write:

$$s = \lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} a_k$$

If the limit $\lim_{n\to\infty} s_n$ does not exist then we say the infinite series $\sum_{k=1}^{\infty} a_k$ diverges.

Geometric Series Test (GST)

If $a_n = a\ r^n$ then $\sum_{n=1}^{\infty} a_n$ is a geometric series which converges only if $|r| < 1$:

$$\sum_{n=0}^{\infty} a\ r^n = a + ar + ar^2 + \cdots = \frac{a}{1-r} \quad , \quad |r| < 1$$

If $|r| \geq 1$ the geometric series diverges.

P-Series Test (PST)

If $p > 1$ the series $\sum \frac{1}{n^p}$ converges. Otherwise the series diverges.

Divergence Test (DT)

If $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Integral Test (IT)

Suppose $a_n = f(n)$ where $f(x)$ is a positive, decreasing, continuous function on $x \geq k$. Then

$$\sum_{n=k}^{\infty} a_n \quad \textit{and} \quad \int_k^{\infty} f(x)dx$$

either both converge or both diverge.

Comparison Test (CT)

$$0 < a_n \le b_n \quad \text{and} \quad \textstyle\sum_{n=1}^{\infty} b_n \text{ converges} \quad \Rightarrow \quad \textstyle\sum_{n=1}^{\infty} a_n \text{ converges}$$

$$0 < b_n \le a_n \quad \text{and} \quad \textstyle\sum_{n=1}^{\infty} b_n \text{ diverges} \quad \Rightarrow \quad \textstyle\sum_{n=1}^{\infty} a_n \text{ diverges}$$

Limit Comparison Test (LTC)

Let a_n and b_n be positive and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

Then the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

When $L = 0, \infty$ we have the following variants:

$$L = 0 \quad \textit{and} \quad \sum b_n \quad \textit{converges} \quad \Rightarrow \sum a_n \quad \textit{converges}$$

$$L = \infty \quad \textit{and} \quad \sum b_n \quad \textit{diverges} \quad \Rightarrow \sum a_n \quad \textit{diverges}$$

Alternating Series Test (AST)

Suppose $\{a_n\}$ is a positive decreasing sequence:

$$\begin{aligned} a_n &> 0 \\ a_{n+1} &\le a_n \\ \lim_{n\to\infty} a_n &= 0 \end{aligned}$$

then the alternating series

$$s = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$$

converges.

When the series converges, the difference between the n^{th} partial sum s_n and s is at most the value of the next term a_{n+1}, i.e.,

$$\left| \sum_{n=1}^{\infty} (-1)^{n-1} a_n - s_n \right| = |s - (a_1 - a_2 + a_3 - a_4 + \cdots + a_n)| \le a_{n+1}$$

Absolute and Conditional Convergence

If $\sum |a_n|$ converges then $\sum a_n$ is said to be absolutely convergent.

If $\sum a_n$ converges but $\sum |a_n|$ diverges then $\sum a_n$ is said to be conditionally convergent.

Every absolutely convergence series is convergent.

Ratio Test (RT)

1) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\sum a_n$ converges absolutely (and hence converges)

2) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ then $\sum a_n$ diverges
$= \infty$ then $\sum a_n$ diverges

3) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \mathrm{L} = 1$ then the test fails and nothing can be said

(RooT) Root Test

1) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ then $\sum a_n$ converges absolutely (and hence converges)

2) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ then $\sum a_n$ diverges
$= \infty$ then $\sum a_n$ diverges

3) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \mathrm{L} = 1$ then the test fails and nothing can be said