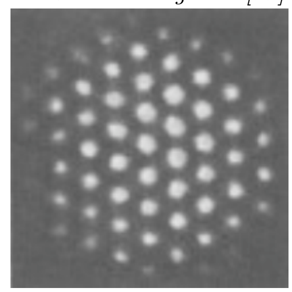
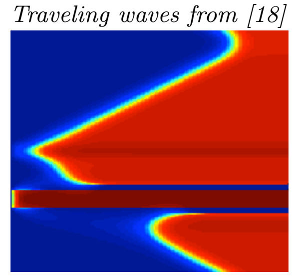
My research combines analysis, computation, and modeling to understand problems stemming from general pattern forming systems. My thesis work applied dynamical systems techniques to understand localized structures in reaction-diffusion systems. Most of these patterns can be understood as stationary or traveling wave solutions to partial differential equations (PDEs). More recently I have focused on social modeling. The patterns in these models range from traveling waves of cooperation in a spatial game to organized crime networks in a game coupled with an evolving social network and are best approached with statistical mechanics techniques. This summer, I also co-directed an REU research project on burglary hotspot modeling.

One of the key approaches I use is spatial dynamics: a spatial coordinate, such as the radius for radially symmetric solutions, is isolated for treatment as an evolution variable. Then invariant manifold techniques in conjunction with perturbative and global dynamical systems methods can be applied to prove the existence of solutions and capture their stability properties. In this setting, a front between a zero rest state and a spatially periodic roll pattern appears as a heteroclinic connection between an equilibrium to a periodic orbit, a simple plateau pulse is a homoclinic orbit from the rest state to itself, and a multi-pulse structure is another homoclinic orbit that makes several rounds in phase space before returning to the rest state. Fronts and pulses are found by tracking the stable and unstable manifolds from the appropriate rest states and seeking intersections; these are not local expansions. Once these are found, more direct approaches are used to glue them together in order to find multi-pulses. With this in mind, a pulse can actually be seen as a front and back glued together and similar techniques apply. Spatial dynamics has proven to be a powerful tool in the study of localized standing and traveling wave solutions.

Gas discharges in [22]

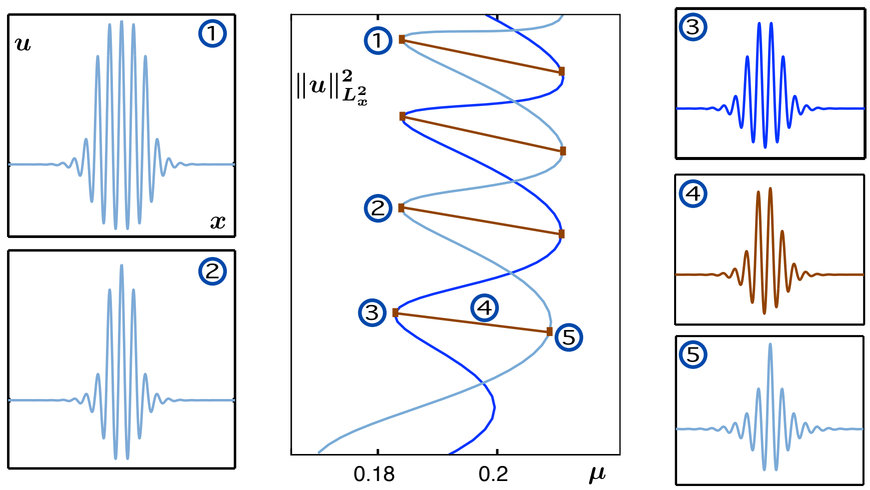


Traveling waves from [18]



Social modeling and adversarial games: My recent research has concentrated on social modeling with an emphasis on new types of game theoretic models. Specifically, the focus has been on including social mechanisms, such as peer pressure or a network of personal connections, in an evolutionary adversarial game introduced in [27] to understand the transition of a disorganized crime dominated society towards peace and cooperation. A population with players of four possible types (witnessing and non-witnessing criminals, and witnessing and non-witnessing non-criminals) undergo pairwise interactions that result in one of the players changing strategy. One of the criminals chooses a victim at random, a crime and possibly a trial occur, and the loser then updates his strategy. In the original paper [27], the authors were also able to derive a system of ordinary differential equations (ODEs) that reproduced the game dynamics for a large number of players. Similarly, my first goal in these extended models is to find a limiting differential equation that I can then study using numerics and dynamical systems. However, some of the models are more conducive to discrete Monte Carlo simulations and random graph theory.

The game, as described above, assumes players will only decide on a strategy as a result of a one-on-one criminal interaction without regard to any neighbor's strategy. However, peer pressure is well known to influence our behavior. Using the ODE formulation of the game, a spatial component can be added and the effects of peer pressure can be included by allowing strategy types to diffuse. In this PDE model, a localized population of witnessing criminals develops into a traveling pulse that mediates the transition from a criminal state to a peaceful one with no criminals. A numerical study has been submitted for publication as [18], and a sample traveling wave can be seen in the above figure. These traveling waves and their selected speed are difficult to analyze, as the invaded state is degenerately unstable. Usually a linear analysis is sufficient to predict such



Displayed are the snaking curves and the associated pulse solutions for the 1D Swift–Hohenberg equation. These curves have been recalculated following [4–6, 8, 32].

traveling waves, but here a fully nonlinear treatment is required because of the degeneracy. I aim to apply techniques from singular perturbation theory, as recently done in [11], to understand both the wave's existence and speed selection rigorously for a reduced two-component model with degenerate traveling waves.

In another variant of the above game, sacred or protected values are introduced as a generating mechanism for criminal coalitions such as gangs and terrorist networks. Sacred values refers to an ideal that an individual views as inviolable. In my case, sacred values refers to a fixed network of personal relationships such as kinship or friendship. Criminals will then choose a victim from outside of their personal network, and those within their personal network will not witness against them. Coalitions can then be defined as the subsets of the criminals that are connected through direct connections on these personal networks. The networks themselves are fixed, but the coalitions can evolve in time. The addition of sacred values in the model stabilizes small amounts of crime and provides protection for the criminal types against witnessing. More details of this work can be found in [19].

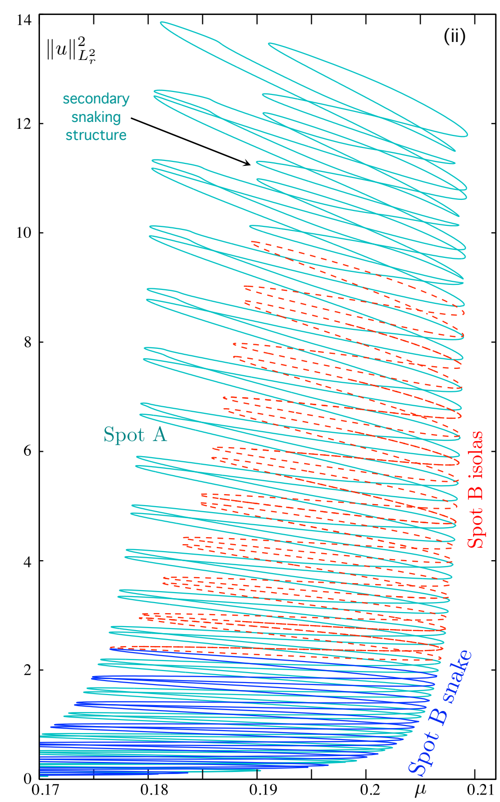
REU mentoring: In the summer of 2012 jointly with Theodore Kolokolnikov, I advised three undergraduate students on a crime modeling project. The model was an adaptation of the burglary hotspot model from [28]. In the model from [28], burglars undergo a biased random walk towards attractive burglary cites. Burglaries are empirically known to be self-exciting: when a house is robbed, it is more likely to be targeted again within a couple of weeks, as are its neighbors. Though this is well understood, a criminal's movements are not. Instead of treating the criminals as random walkers, we modeled their movements with a Lévy flight. This captures the different modes of transportation a criminal could access such as cars, trains, and walking. Mathematically, this leads to a nonlocal equation for the burglary hotspots. We were able to derive the nonlocal limiting equation, numerically solve the agent based system and the nonlocal equation, perform a linear stability analysis to predict the development of crime hotspots, as well as an inner-outer matching procedure in a singular limit to predict the hotspot profile. A journal publication has been submitted [7].

Pattern formation: Patterns constantly appear in nature. Localized spots and concentric rings are seen in desert grasses in the Negev, as burglary hotspots, as oscillons in vertically vibrated trays of sand and clay, and as standing structures in basins of ferromagnetic fluids in a strong magnetic field. These spots are seen organized into hexagon patches in gas discharge experiments, chemical reactions (CIMA), and liquid crystal displays ([16], [1] and references therein). The Swift–Hohenberg equation,

$$u_t = -(1+\Delta)^2 u - \mu u + \nu u^2 - \kappa u^3 + \mathrm{O}(u^4), \qquad x \in \mathbb{R}^n,$$

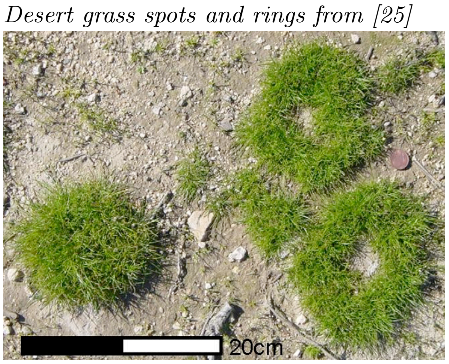
is studied as an archetypical system to understand the formation of these patterns. Originally derived in [29] to understand the onset of convective rolls from thermal fluctuations in fluid systems, it supports a gallery of spatially localized solutions. For example, the planar equation supports a plethora of radially symmetric spot and ring solutions as well as hexagon patches. It also can be seen as the normal form for Turing (or pattern forming) bifurcations in planar reaction-diffusion systems as was shown in [24]. The figures throughout this research statement illustrate a zoo of experimental patterns, some of which are well modeled by Swift–Hohenberg. The Swift–Hohenberg equation is interesting both physically and mathematically for dimension $n = 1, 2, 3$. On the domain $\mathbb{R}$, it exhibits snaking. This phenomenon is characterized by an infinite number of solutions, in this case the localized roll patterns, existing for a fixed value of the bifurcation parameter μ. These solutions are all connected in the bifurcation diagram by a single curve that looks like a snake's tracks, as is seen in the figure on page 2. While conditions for snaking and the existence of the accompanying pulse solutions have been studied extensively, numerous issues related to this equation are wide open for exploration.

Terminated Snaking: By concentrating on radially symmetric rolls, I used numerical continuation techniques to look at the snaking curves in 2 and 3D. The study of radially symmetric solutions can be reduced to a one-dimensional problem, but the dimension still appears in the radial laplacian. I treated the dimension n as a continuous parameter to understand the transition away from snaking. Surprisingly, the snaking appears to terminate for $n > 1$; these results are published in [17]. The bifurcation curves are shown to the right. Above these initial branches are a set of stacked isolas and a high snaking branch which exhibits defect mediated snaking. This behavior persists into three dimensions, but the extent of the snaking branches and stacks of isolas varies. Lloyd and Sandstede found two one-parameter families of ring solutions and one family of spot solutions, spot A, in 2D, and rigorously established their existence at onset in [15]. As a result of the terminated snaking, I found a second family of spots, referred to as spot B. Because the dimension was treated in a continuous fashion, I was able to follow the planar solutions back into one dimension. This revealed that the spots in 2D went to snaking spots in 1D, whereas the rings became 2-pulses. In bifurcation space, these exist along figure-8 isolas lying on top of the snaking branches and accompanying ladders. These isolas combine through a series of saddle-node bifurcations to become continuous curves in 2D. The spot B family of solutions behaves differently than the rings or spot A, and a different analysis from [15] is required to understand their existence.

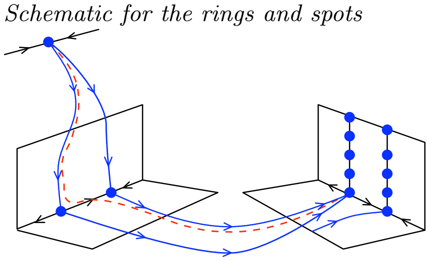


Existence: In 2D, Spot B and the rings have numerically been seen to exist only for $\mu > \sqrt{27/38}$ while spot A has been found below this. However, the amplitude of spot B scales differently than the rings or spot A when the bifurcation parameter μ is reduced to zero. The amplitude of spot A and the rings varies like $\sqrt{\mu}$. In contrast, for spot B this was numerically seen as $\mu^{0.374}$. The $\sqrt{\mu}$ scaling is natural for these solutions and it is hard to imagine where the $\mu^{0.374}$ comes from. In the schematic for stationary solutions of the planar Swift–Hohenberg equation, three different coordinate charts for the amplitude equations are displayed: the top left is the core, the bottom left is the transition region which captures algebraic growth and decay, and the far right is the far field which captures exponential growth and decay. The rings and spot A were found by looking at heteroclinic connections between the core (where, in this spatial dynamical setting, the core captures solutions that are bounded and smooth at the origin) and the equilibria in the transition chart and then heteroclinic connections between these equilibria and the far field equilibria; the rings and spot A are found by gluing these connections together. When searching for solutions analytically, these equilibria and the corresponding connections are all that is usually studied. The scaling is from the far field amplitude equations, a complex Ginzburg–Landau equation in 2D, which require a $\sqrt{\mu}$ scaling.

Desert grass spots and rings from [25]



Schematic for the rings and spots



To find spot B, I needed to proceed differently. A center manifold reduction was used to reduce to the amplitude equations wherein the solutions are combinations of Bessel functions. In the transition regime, I used geometric blow-up techniques. Assuming the solution is of the form $u = r^{\alpha}$, then transforming into the coordinates $v := \frac{ru_r}{u} = \alpha$ the algebraic growth or decay rate can be captured. To prove spot B's existence, the solutions had be tracked in each coordinate chart and then combined using matched asymptotics. In order to track the solutions adequately around each equilibrium in the transition chart, normal forms were used. Then, by showing the solution operators were contractions in appropriate function spaces, the spot B solution was pieced together. Spot B's transit between the two equilibria conspires to produce the scaling $\mu^{\frac{3}{8}}$, which is in very good agreement with the numerics. The same techniques can be used to rigorously establish the existence of both spots A and B in three dimensions. In the core, the solutions become sines and cosines modulated by the radius, rather than Bessel functions. Additionally, one of the equilibria in the transition chart loses hyperbolicity. The proofs still work, however. See [21] for details. These techniques can be applied to study oscillons ([30] and [33]) in the forced complex Ginzburg-Landau equation, and a coupled Turing-Hopf system. Kelly McQuighan and Bjorn Sandstede have already began work on the forced Ginzburg-Landau system.

Stability: An unresolved aspect of the 1D Swift–Hohenberg equation is the stability of symmetric and asymmetric pulses. The spectrum determines the linear stability of these pulses, and is the sum of the continuous spectrum and point spectrum (isolated eigenvalues). The functions associated with the continuous spectrum are not localized in space, unlike eigenvalues. The essential spectrum only depends on the asymptotic rest states and is relatively straightforward to find. Once the spectrum is known, it is easy to assess stability. Any spectrum with positive real part causes an instability. These can be related to solutions breathing, traveling, blowing up, or decaying to zero. The symmetric pulses alternate between stability and instability at each fold in the bifurcation diagram, and the dominate mode associated with the fold is localized at the edge of the snaking patterns. The numerically computed eigenvalues from [3] behave in a curious manner along the snaking curve: the saddle-node eigenvalue appears to pass into the continuous spectrum. This is surprising because, in a similar situation, [12] found this was impossible.

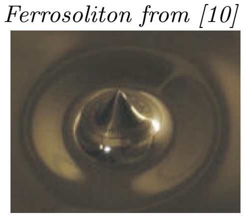
In 2D, both spots numerically appear to be stable under radially symmetric perturbations. As the planar patterns become broad enough, the radial equation formally reduces to the 1D equation (as $r \to \infty$) at the edge of the pattern. The dominate unstable mode then approaches the localized unstable mode from 1D. Under arbitrary perturbations however, spot A and B are both unstable. Spot A is known to undergo a symmetry breaking bifurcation that forms hexagons [16], but the behavior of spot B is unknown. I am presently studying the dynamics for these spots in both two and three dimensions.

The techniques I used to prove the existence of spots in 2D and 3D are also applicable to study the stability of these solutions. I want to examine the spectrum of the linearization of the Swift–Hohenberg equation around the spots. Using an appropriate ansatz, the eigenvalues can be found by solving the original equation with one additional term. I can then use the same charts and matching procedure as was used to prove existence to find the eigenvalues, though the analysis is much more involved.

Heavy-ion fusion-fission physics: Early in graduate school, I studied nuclear reaction theory under the supervision of Dr. John Lestone at Los Alamos National Laboratory. In heavy-ion collisions, a large nucleus, such as oxygen, is accelerated and collided with an enormous nucleus, such as uranium. The two nuclei fuse into an extremely excited state. Energy is then dissipated through ejecting gamma rays, neutrons, and charged particles, and eventually fissioning. It was believed that an energy dependent nuclear viscosity was required to accurately predict the neutron emission and fission rates. This theory disagreed with direct measurements of the nuclear viscosity, however. We were able to show that an energy dependent nuclear viscosity was unnecessary to reproduce the data through an improved model and systematic Monte Carlo simulations. This resulted in two publications [14, 20].

Future directions: Many pattern forming systems exhibit low-dimensionality in their solutions, even though the models are infinite dimensional. As an example, the 1D snaking solutions for the Swift–Hohenberg equation all differ by a single localized roll that remains unchanged as one progresses up the bifurcation curve. In [9, 26, 31], a Proper Orthogonal Decomposition (POD) has been performed on the solutions to several PDEs in order to extract low-dimensional structures in the solutions. These modes are then used to calculate the bifurcation curves, often with massive improvements in computational cost but no loss in fidelity. Nathan Kutz has proposed studying the 2D snaking curves using a POD, adding a mode after every fold. Hopefully we would be able to extract information on why snaking terminates by examining the small changes in the dominate modes as we travel up the bifurcation curve.

Ferrosoliton from [10]



Hexagons from [23]

Three-dimensional Turing patterns have been experimentally found in chemical reactions; see [2, 13]. They first used a PDE solver for a two component model equation to predict the existence of 3D Turing spots, and then used tomograpy to image the corresponding chemical reaction as it progressed. It would be interesting to apply the same methodology used for the Swift–Hohenberg equation, which is the normal form for Turing bifurcations in reaction-diffusion equations, to their model equations.

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